Finding the greatest and least values of an algebraic function on a given interval

Types of intervals: Open Interval: eg $-3 < x < 3$ means we **exclude** the end points

Closed Interval: eg $-3 \leq x \leq 3$ means the end points are **included**

To find maximum and minimum values on a given interval, we need to check **stationary points** as before, but also consider the **end points** of the interval

eg: Find the maximum and minimum values of $f(x) = x3 - 3x + 2$ on the interval $-2 \leq x \leq 3$

x^3

$f(x) = x^3 - 3x + 2.$

$f'(x) = 3x^2 - 3$
$= 3(x^2 - 1)$
$= 3(x-1)(x+1)$

For stationary points $\quad f'(x) = 0$
$3(x-1)(x+1) = 0$
$x = 1$ or $x = -1$

y co-ords $\quad x = 1 \Rightarrow f(1) = 1 - 3 + 2$
$= 0$ $\quad (1,0)$

$x = -1 \Rightarrow f(-1) = -1 + 3 + 2$
$= 4$ $\quad (-1,4)$

nature

x	$\xrightarrow{(-2)}$	-1	$\xrightarrow{(0)}$	1	$\xrightarrow{(2)}$
$f'(x) = 3(x-1)(x+1)$	9	0	-3	0	9
shape	/	—	\	—	/

$(1,0)$ is a minimum turning point
$(-1,4)$ is a maximum turning point.

End point values $\quad x = -2 \quad f(-2) = (-2)^3 + 6 + 2.$
$= 0$

$x = 3 \quad f(3) = 3^3 - 9 + 2.$
$= 20$

On the interval minimum value $= 0$
maximum value $= 20.$

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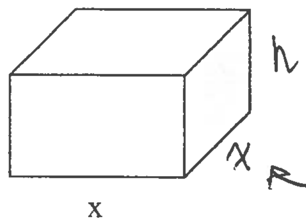
Optimization

We can find maximum and minimum values of things using stationary points.
Example

1) The volume of the square-based display case shown is 500cm³

   The length of the base is x cm. The base of the case is not made of glass.



let height $= h$.

since square based

a) Show that the area of glass in the case is $A(x) = x^2 + \dfrac{2000}{x}$
b) Find the dimensions of the case which minimises the use of glass and calculate this minimum area.

(a) Area of glass $A(x) = \underbrace{4xh}_{\text{sides}} + \underbrace{x^2}_{\text{top}}$ (*)

We want the formula to contain only one letter (x)
Use what we know ie $V = 500\text{cm}^3$ to get h in terms of x.

$$V = lbh$$
$$500 = x \times x \times h$$
$$h = \frac{500}{x^2}.$$

In (*) $A(x) = 4x \times \dfrac{500}{x^2} + x^2.$

$$= \frac{2000}{x} + x^2.$$

$$= x^2 + \frac{2000}{x} \text{ as required.}$$

PTO for (b)

(b) * They always give you the formula in part (a) so that you can still do part (b) even if you can't do part (a) * *

$A(x) = x^2 + 2000x^{-1}$

$A'(x) = 2x - 2000x^{-2}$

$= 2x - \frac{2000}{x^2}.$

For minimum area. $A'(x) = 0$

$2x - \frac{2000}{x^2} = 0$

$2x^3 - 2000 = 0$

$x^3 = 1000$

$x = 10$

Nature.

x	(i) $\rightarrow$	10	(ii) $\rightarrow$
$\frac{dA}{dx} = 2x - \frac{2000}{x^2}$	-1998	0	5.5
	$-$	0	$+$
shape.	\	—	/

So $x = 10$ gives a minimum area.

When $x = 10$ $h = \frac{500}{10^2}$

$= 5$

The dimensions of the case which give a minimum area are 10cm by 10cm by 5cm

make sure you answer the question properly

$A(x) = x^2 + \frac{2000}{x}$

$A(10) = 100 + \frac{2000}{10} = \underline{\underline{300cm^2}}.$

Solving problems involving rates of change

Examples

1) The pulse rate in beats per minute of a runner **t** minutes after starting to run is given by

$$\mathbf{p(t) = 60 + \tfrac{1}{2}t^2 - t \qquad \text{for } t < 10}$$

Find the rate of change of the pulse of the runner 6 minutes after starting.

↳ differentiate ↑ put $t=6$

$p'(t) = t-1$
$p'(6) = 6-1$
$= 5.$

2) The velocity of a particle in meters per second, after time t seconds, moving in a straight line along the x-axis is given by $v(t) = t - \sqrt{t}$, $t > 0$. Find the acceleration of the particle after 4 seconds.

* In Unit 2 NAB !!

displacement $s(t), d(t)\ h(t)$ $\xrightarrow{\text{differentiate}}$ velocity $v(t)$ $\xrightarrow{\text{differentiate}}$ acceleration $a(t)$.

$v(t) = t - \sqrt{t}$
$= t - t^{\frac{1}{2}}$

$a(t) = v'(t)$
$= 1 - \frac{1}{2}t^{-\frac{1}{2}}$
$= 1 - \frac{1}{2\sqrt{t}}$

When $t=4$ $a(4) = 1 - \frac{1}{2\sqrt{4}}$
$= \frac{3}{4}\ m/s^2$

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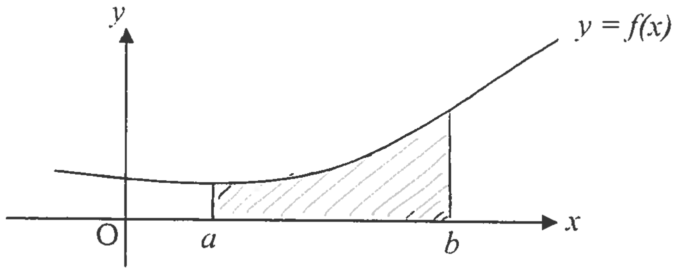
Area between a curve and the x-axis



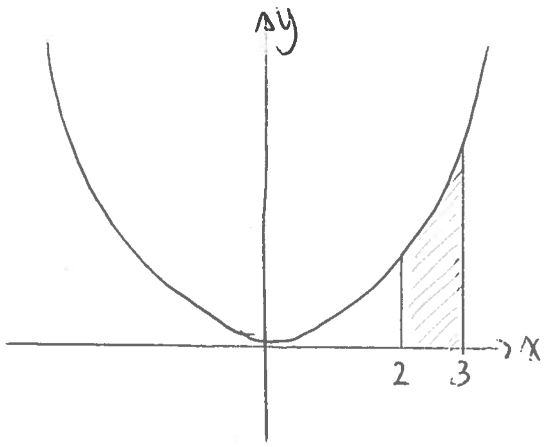
The shaded area between a line or a curve y = f(x) and the x-axis between the values x = a and x = b is given by

$$\int_a^b f(x)dx$$

Examples

1) Calculate the area between the curve $y = 3x^2$ and the x-axis between x = 2 and x = 3

* Make a sketch first



$$\text{area} = \int_2^3 3x^2\,dx$$

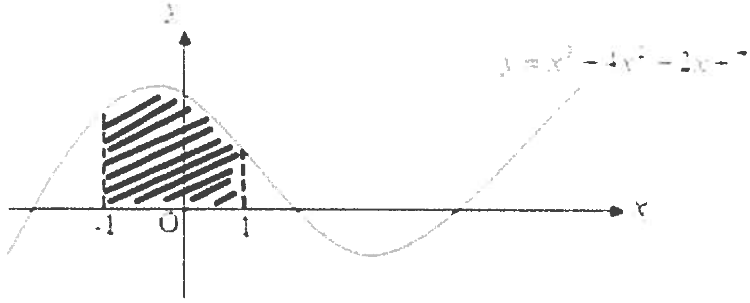
$$= \left[x^3\right]_2^3$$

$$= 3^3 - 2^3$$

$$= 27 - 8$$

$$= 19 \text{ square units.}$$

2) Calculate the shaded area shown in the diagram



area $= \int_{-1}^{1} (x^3 - 4x^2 - 2x + 7)\,dx$

$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - x^2 + 7x \right]_{-1}^{1}$

$= \left(\frac{1}{4} - \frac{4}{3} - 1 + 7 \right) - \left(\frac{1}{4} + \frac{4}{3} - 1 - 7 \right)$

$= -\frac{8}{3} + 14$

$= 11\frac{1}{3}$

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Areas Above and Below the x-axis

When calculating using integration areas below the x-axis give a negative value and areas above the x-axis give a positive value.

So for an area below the x-axis

$$\text{Area} = -\int_a^b f(x)\,dx$$

We calculate the areas above and below separately and then add them.

- Check limits – find them if not given
- Draw a sketch
- Areas below and above x-axis do separately
- LIMITS $\rightarrow$ SKETCH $\rightarrow$ INTEGRATE

Examples

1) Calculate the area bounded by the curve $y = x^2 - 6x + 8$ and the axes.

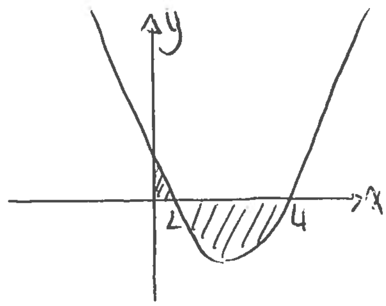
$y = x^2 - 6x + 8$ parabola $+x^2$ U

cuts x-axis

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2$ or $x = 4$



area above x-axis

$$\int_0^2 (x^2 - 6x + 8)\,dx$$

$$= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2$$

$$= \left(\frac{8}{3} - 12 + 16\right) - 0$$

$$= 6\tfrac{2}{3}$$

area below x-axis

$$-\int_2^4 (x^2 - 6x + 8)\,dx$$

$$= -\left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4$$

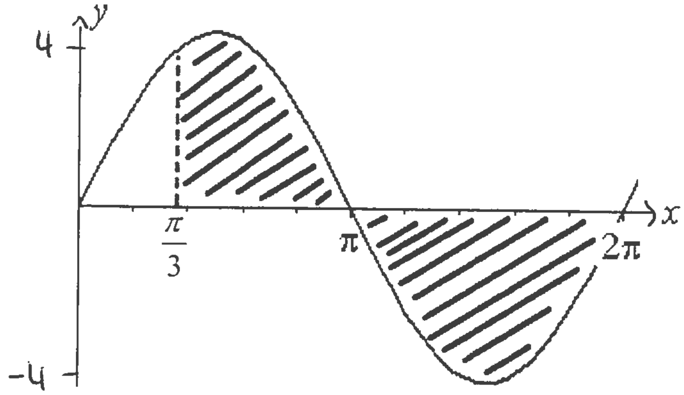
$$= -\left(\left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right)\right)$$

$$= 1\tfrac{1}{3}$$

total area $= 6\tfrac{2}{3} + 1\tfrac{1}{3}$

$= 8$ square units

2) Part of the graph of y = 4 sinx is shown. Calculate the shaded area.



area above

$$\int_{\frac{\pi}{3}}^{\pi} 4\sin x \, dx$$

$$= \left[-4\cos x\right]_{\frac{\pi}{3}}^{\pi}$$

$$= -4\cos\pi + 4\cos\frac{\pi}{3}$$

$$= 4 + 4 \times \frac{1}{2}$$

$$= 6$$

area below

$$-\int_{\pi}^{2\pi} 4\sin x \, dx$$

$$= -\left[-4\cos x\right]_{\pi}^{2\pi}$$

$$= -\{-4\cos 2\pi + 4\cos\pi\}$$

$$= 4\cos 2\pi - 4\cos\pi$$

$$= 4 + 4$$

$$= 8$$

total area = 6+8

= 14 square units.

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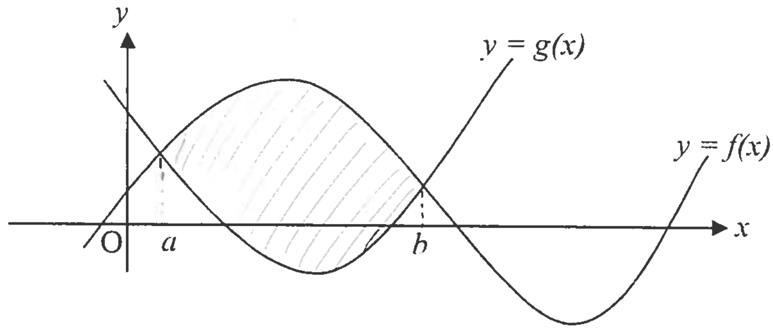
Area Between Two Curves



Shaded Area = $\int_a^b (f(x) - g(x))dx$ where $f(x) \geq g(x)$ for $a \leq x \leq b$

(i.e. **upper curve – lower curve**)

Example

Calculate the area enclosed by the curve $y = x^2 - 4$ and the line $y = 2 - x$

* Must make a sketch if not given *

Remember : –

- If limits are not given you must calculate them
- Limits are x-coordinates of points of intersection

Limits → points of intersection

solve $y = x^2 - 4$ and $y = 2 - x$

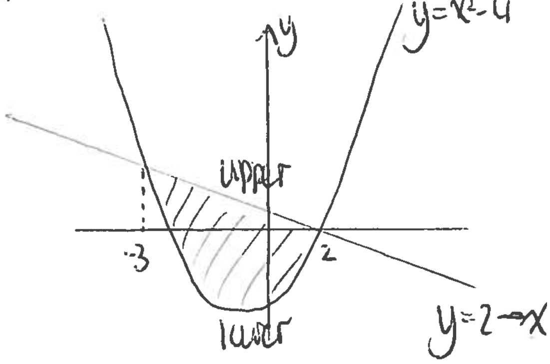
$x^2 - 4 = 2 - x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ $\quad$ $x = 2$

$y = 5$ $\quad$ $y = 0$



area $= \int_{-3}^{2} (\text{upper} - \text{lower})\, dx$

$= \int_{-3}^{2} ((2-x) - (x^2-4))\, dx$

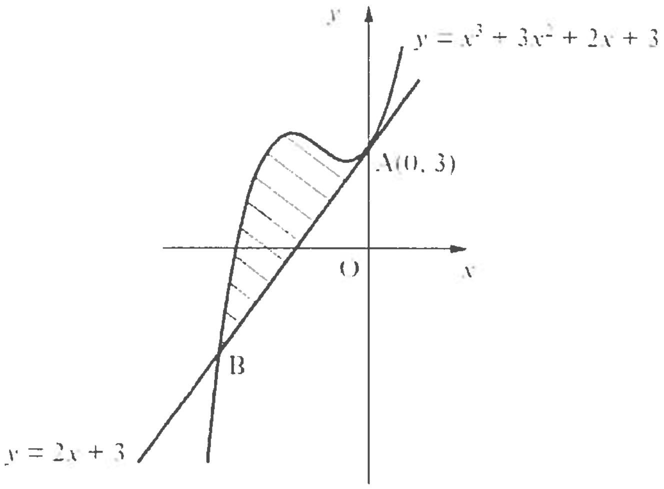
$= \int_{-3}^{2} (6 - x - x^2)\, dx$ ✓ Simplify first then integrate

$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-3}^{2}$

$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} - \left(\frac{-27}{3}\right)\right) = 20\frac{5}{6}$

Note For areas between curves it doesn't matter if some of the area is below x-axis

2) The line with equation $y = 2x + 3$ is a tangent to the curve with equation $y = x^3 + 3x^2 + 2x + 3$ at A(0, 3), as shown in the diagram.



The line meets the curve again at B.

Show that B is the point (−3, −3) and find the area enclosed by the line and the curve.

Solve $\left.\begin{array}{l} y = 2x+3 \\ y = x^3+3x^2+2x+3 \end{array}\right\}$

$x^3+3x^2+2x+3 = 2x+3$

$x^3+3x^2 = 0$

$x^2(x+3) = 0$

$x = 0$ or $x = -3$

$y = 3$ $\quad y = -6+3 = -3.$ $\quad$ B(−3, −3) as required.

area $= \int_{-3}^{0} (\text{upper} - \text{lower})\, dx$

$= \int_{-3}^{0} \left((x^3+3x^2+2x+3) - (2x+3)\right) dx$

$= \int_{-3}^{0} (x^3+3x^2)\, dx$

$= \left[\frac{x^4}{4} + x^3\right]_{-3}^{0}$

$= 0 - \left(\frac{(-3)^4}{4} + (-3)^3\right)$

$= 6\frac{3}{4}$ square units.

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