AS 1.3 Applying Algebraic Skills To Sequences

Recurrence Relations

A recurrence relation describes a sequence where each term is a function of the previous term. A recurrence relation will generate different sequences for different starting values.

Compound Interest is an example of a recurrence relation.

u_0 is the initial value

u_n is value after n years

e.g. £32500 initial investment with 2.5 % interest per year

2.5% added
$\Rightarrow$ 102.5% altogether
$= 1.025$
multiplier

$$u_0 = £\,32500$$
$$u_{n+1} = 1.025u_n$$

> A sequence defined by $\mathbf{u_{n+1} = au_n + b,\ a \neq 0}$ is called a **linear** recurrence relation.

Examples

1) Find a recurrence relation, including the first term u_1 for the sequence 7, 11, 15, 19,..

$$7,\ 11,\ 15,\ 19$$
$$+4 \quad +4$$
$$u_{n+1} = u_n + 4 \qquad u_1 = 7$$

2) A sequence is defined by the recurrence relation $\mathbf{u_{n+1} = 0.6u_n + 5,\ u_0 = 8.}$
Calculate the value of $\mathbf{u_3}$ and find the smallest **n** such that $\mathbf{u_n > 12}$

$$u_1 = 0.6u_0 + 5$$
$$= 0.6 \times 8 + 5$$
$$= 9.8$$
$$u_2 = 0.6 \times 9.8 + 5$$
$$= 10.88$$
$$u_3 = 0.6 \times 10.88 + 5$$
$$= 11.528$$

$u_4 = 11.9168..$

$u_5 = 12.15...$

The smallest value of n such that $u_n > 12$ is $n = 5$

p321 Ex 15B no 1
Ex 15C 1 2 4

3) At the start 2001 the population in India was 586 million. If the annual rate of population increase is 2 % what will the population be by the end of 2004 ? Find a recurrence relation for the population growth ?

2% increase ⇒ 102% altogether
= 1.02.

4 years growth 586×1.02^4 million
= 634.3 million (4 s.f)
at end of 2004.

$u_{n+1} = 1.02u_n$ where u_n = population at the start of the nth year after 2001.

$u_0 = 586$ million.

4) Supposing with example 3 we take into consideration that 1.5 million die every year. Construct a new recurrence relation and work out a more accurate answer for the population in 2004.

$u_{n+1} = 1.02u_n - 1.5$

Population at end of 2004 is 4 years after start of 2001.

ie find u_4.

$u_1 = 1.02 \times 586 - 1.5$
$= 596.22$.

$u_2 = 1.02 \times 596.22 - 1.5$
$= 606.6444$

$u_3 = 617.277288$

$u_4 = 628.1228338$ 628.1 million (4 s.f)

(5) In a park 90% of the litter is removed each week and each week the public drop 25kg of litter. Make a recurrence relation. assuming there is no litter to start with.

Let u_n = amount of litter after nth week.

$u_0 = 0$

90% removed ⇒ 10% left
= 0.1

$u_{n+1} = 0.1u_n + 25$

% multiplied amount added.

Ex 15B, pg 321 -322 No (2) → (5)
Recurrence Booklet Ex 15C (3) (

AS 1.3 Applying Algebraic Skills To Sequences

Solving Recurrence Relations - $u_{n+1} = au_n + b$ - to find a and b

Example

A recurrence relation is defined by $u_{n+1} = au_n + b$
If $u_1 = 5$, $u_2 = 9.5$ and $u_3 = 20.75$ find the values of a and b.

$u_{n+1} = au_n + b$

$u_2 = au_1 + b \Rightarrow 9.5 = 5a + b \quad \ldots (1)$

$u_3 = au_2 + b \Rightarrow 20.75 = 9.5a + b \quad \ldots (2)$

Solve simultaneously

$(2) - (1) \quad 4.5a = 11.25$

$a = 2.5$

in $(1) \quad 5 \times 2.5 + b = 9.5$

$12.5 + b = 9.5$

$b = -3$

So $a = 2.5$, $b = -3$.

Ex 15C, pg 323 - 324
Q 6 7.

The Limit of a Recurrence Relation

For a linear recurrence relation $\mathbf{u}_{n+1} = \mathbf{a}\mathbf{u}_n + \mathbf{b}$ if $\mathbf{-1 < a < 1}$ then $\mathbf{u}_n$ tends to a limit as $n \to \infty$ and the limit $L = \dfrac{b}{1-a}$, and we say that the recurrence relation is **convergent.**

Examples

1) Find the first three terms and the limit if it exists of the recurrence relation

$\mathbf{u}_{n+1} = \mathbf{0.6u}_n + \mathbf{12}, \ \mathbf{u}_0 = \mathbf{20}$

$u_0 = 20$
$u_1 = 0.6 \times 20 + 12$
$= 24$
$u_2 = 0.6 \times 24 + 12$
$= 26.4$

Limit exists since $-1 < 0.6 < 1$

You must state this before finding a limit

$L = \dfrac{b}{1-a}$
$= \dfrac{12}{1-0.6}$
$= \dfrac{12}{0.4}$
$= 30.$

2) A forestry company owns 100 hectares of woodland. Each hectare contains 60 mature trees. To maintain stocks the company needs to have 1500 mature trees at any one time. It estimates that 25% of mature trees are felled each month while 300 trees reach maturity over the same period of time. Will stocks be maintained at an acceptable level?

Use this information to find a recurrence relation.

consider limit to determine what happens in the long term

Number of trees to start with $= 100 \times 60$
$= 6000$

Let u_n = number of trees at the end of each month.

$25\% \Rightarrow 75\%$ altogether
$= 0.75$

so $u_{n+1} = 0.75u_n + 300$

Since $-1 < 0.75 < 1$ limit exists

$L = \dfrac{b}{1-a}$
$= \dfrac{300}{1-0.75}$
$= 1200$

Since limit is less than 1500 the stocks will not be maintained at an acceptable level.

3) Secret Agent 004 has been captured and his captors are giving him a 25mg dose of truth serum every four hours. 15% of the truth serum present in his body is lost every hour.

a. Calculate how many mg of serum are present in his body after 4 hours (just before the second dose is given)
b. Let u_n be the amount of serum in the body just after the nth dose. Determine a recurrence relation for u_n
c. 55mg of serum in the body would prove fatal and his captors wish to keep him alive. Can the captors safely continue this treatment?

(a) 15% lost each hour ⇒ 85% left $= 0.85$

amount after 4 hours $= 0.85^4 \times 25$
$= 13.1\text{mg}$ (3 s.f).

(b) percentage multiplier. 0.85^4 ***

amount is added every 4 hours so % multiplier must be for 4 hours too.

$= 0.5220$ (4 s.f)

$u_{n+1} = 0.5220u_n + 25$

(c) Find limit

Since $-1 < 0.5220 < 1$ a limit exists

$$L = \frac{b}{1-a} = \frac{25}{1-0.5220} = 52.3\text{mg} \quad (3\text{ s.f})$$

Yes it should be safe to continue the treatment because in the long term he will have 52.3mg of serum in his body after each dose which is less than 55mg.

Ex15D, pg 326

Harder Example (Recurrence Booklet Ex 3A, Q3)