The Equation Of A Circle

Not done!!

Recall the **distance formula**

Two points (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

or

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

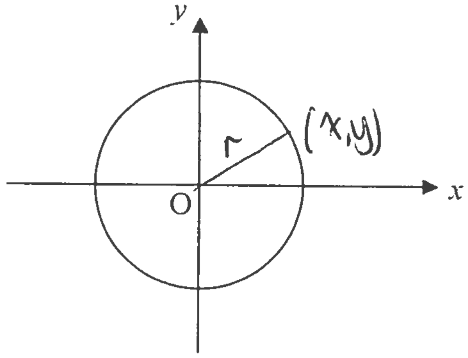
From this we get the **equation of a circle**

(i) Circle centre (0,0) and radius r $\qquad$ $\mathbf{x^2 + y^2 = r^2}$ (must know)

(ii) Circle centre (a,b) and radius r $\qquad$ $\mathbf{(x-a)^2 + (y-b)^2 = r^2}$ (given)

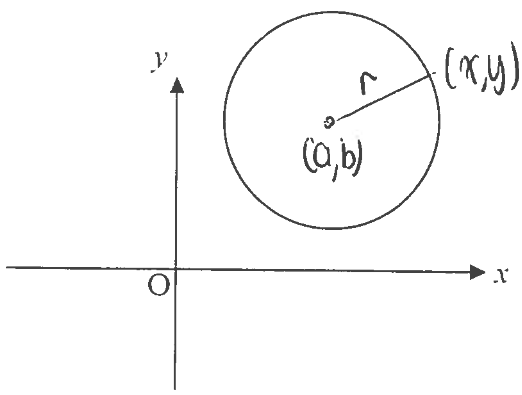
* Always use this equation when asked to find the equation of a circle.

(i)



This circle is all points P(x,y) such that the distance from (0,0) is r
i.e. $r^2 = (x-0)^2 + (y-0)^2$
$r^2 = x^2 + y^2$

(ii)



This circle is all the points P(x,y) such that the distance from (a,b) is r
i.e.
$r^2 = (x-a)^2 + (y-b)^2$

To find the equation of a circle we need :-

- **the centre of the circle**
- **the radius of the circle**

Examples

1) Write down the equations of the circles

a) centre origin, radius $\frac{3}{2}$

b) centre (6, -2), radius 3

(a) $(x-a)^2+(y-b)^2=r^2$

$(x-0)^2+(y-0)^2=\left(\frac{3}{2}\right)^2$.

$x^2+y^2=\frac{9}{4}$

(b) $(x-a)^2+(y-b)^2=r^2$

$(x-6)^2+(y+2)^2=3^2$

$(x-6)^2+(y+2)^2=9$

NB Don't multiply out brackets unless you need to.

2) Describe the circle with equation

a) $x^2 + y^2 = 64$

b) $(x - 4)^2 + (y + 3)^2 = 25$

(a) centre $(0,0)$ radius $\sqrt{64}$ $=8$

(b) centre $(4,-3)$ radius $\sqrt{25}$ $=5$

3) Find the equation of the circle passing through the point (6,3) and has centre (3,1)

$r=\sqrt{(6-3)^2+(3-1)^2}$

$=\sqrt{9+4}$

$=\sqrt{13}$

$(x-a)^2+(y-b)^2=r^2$

$(x-1)^2+(y-3)^2=13$.

Ex 14A, pg 308 – 310

The General Equation Of A Circle

Lets take an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ and complete the squares in x and y:

$$x^2 + 2gx + y^2 + 2fy + c = 0$$

$$(x+g)^2 - g^2 + (y+f)^2 - f^2 + c = 0$$

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

Compare with

$$(x-a)^2 + (y-b)^2 = r^2$$

Use to find centre and radius of circle if given in multiplied out form.

We can see that $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$ represents a circle with centre $(-g, -f)$ and radius, $r = \sqrt{g^2 + f^2 - c}$, provided $g^2 + f^2 - c > 0$. This is known as the **general equation of a circle.**

* On * Formula sheet

Examples

1) Describe the circle with equation $x^2 + y^2 - 4x + 8y - 16 = 0$

$2g = -4$ $2f = 8$ $c = -16$

$g = -2$ $f = 4$

centre $(2, -4)$

radius $\sqrt{(-2)^2 + 4^2 - (-16)}$

$= \sqrt{36} = 6$

2) State whether $x^2 + y^2 - 6x + 2y - 14 = 0$ represents the equation of a circle

$2g = -6$ $2f = 2$ $c = -14$

$g = -3$ $f = 1$

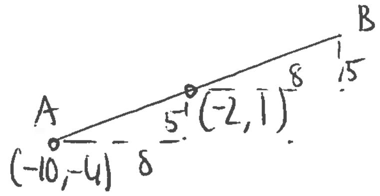
$g^2 + f^2 - c = (-3)^2 + 1^2 - (-14)$

$= 24 > 0$ so it does represent a circle.

3) A circle $x^2 + y^2 + 4x - 2y - 84 = 0$ has diameter AB. If A is the point (-10,-4), find the coordinates of B.

$2g = 4$ $2f = -2$

$g = 2$ $f = -1$ centre $(-2, 1)$



$B(-2+8, 1+5)$

$B(6, 6)$

4) Show that (5, $\sqrt{11}$) lies on the circle centre origin and radius 6

Circle $(x-a)^2+(y-b)^2=r^2$

$(x-0)^2+(y-0)^2=6^2$

$x^2+y^2=36.$

Substitute $(5, \sqrt{11})$ into LHS.

$5^2+(\sqrt{11})^2$

$=25+11$

$=36$ so point lies on circle.

$=$ RHS

5) Does the point (-7, -9) lie inside, outside or on the circle centre (-3,-5) and radius $\sqrt{30}$?

distance from $(-7,-9)$ to $(-3,-5)$

$d=\sqrt{(-7+3)^2+(-9+5)^2}$

$=\sqrt{16+16}$

$=\sqrt{32}$

$>\sqrt{30}$ so point lies outside circle.

Ex 14B, pg 312 – 313

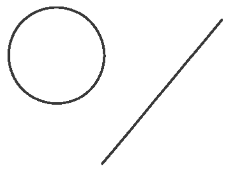
Intersection Of A Line And A Circle

To find the points of intersection we solve simultaneously the equation of the circle and the equation of the straight line. This results in a quadratic equation.

As with a curve and a straight line, a circle and a straight line can have

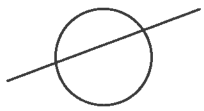
For resultant quadratic

(i) no points of contact



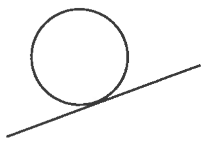
$b^2 - 4ac < 0$

(ii) two points of contact



$b^2 - 4ac > 0$

(iii) one point of contact
(line is a tangent)



$b^2 - 4ac = 0$

Examples

1) Find the points of contact of the line $y = 2x + 8$ and the circle $x^2 + y^2 + 4x + 2y - 20 = 0$

$x^2+y^2+4x+2y-20=0$ .... ①

$y=2x+8$ .... ②

Substitute ② in ①

$x^2+(2x+8)^2+4x+2(2x+8)-20=0$

$x^2+4x^2+32x+64+4x+4x+16-20=0$

$5x^2+40x+60=0$

$x^2+8x+12=0$

$(x+6)(x+2)=0$

$x=-6 \qquad x=-2$

in ② $\qquad y=-4 \qquad y=4$

points $(-6,-4)$ $(2,4)$

2) Show that **$3x + y = -10$** is a tangent to the circle **$x^2 + y^2 - 8x + 4y - 20 = 0$** and find the point of contact.

Solve $x^2+y^2-8x+4y-20=0$ ... ①
$3x+y=-10$ ... ②

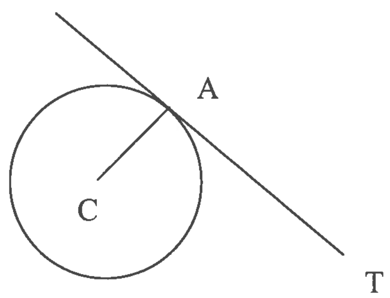
② gives $y=-3x-10$
In ① $x^2+(-3x-10)^2-8x+4(-3x-10)-20=0$
$x^2+9x^2+60x+100-8x-12x-40-20=0$
$10x^2+40x+40=0$
$x^2+4x+4=0$
$(x+2)(x+2)=0$
$x=-2$ equal roots so line is a tangent

Put $x=-2$ in ② $y=6-10$
$=-4$

$(-2,-4)$

p315 Ex 4C
Q ⑤ → ⑧.

Equation Of A Tangent At A Point On The Circle



Equation of tangent AT need (i) point on the line (ii) gradient of line
A is point on the line and $m_{AT} \cdot m_{AC} = -1$ since the tangent is perpendicular to the radius AC

Example

1) Show that A(7,5) lies on the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ and find the equation of the tangent at A.

A(7,5)
↑ x ↑ y

substitute in $x^2+y^2-6x-4y-12$
$= 7^2+5^2-42-20-12$
$= 0$ so (7,5) lies on circle.

Equation of tangent

$x^2+y^2-6x-4y-12=0$
$2g=-6$ $2f=-4$
$g=-3$ $f=-2$ centre (3,2)

m radius $= \frac{5-2}{7-3}$
$= \frac{3}{4}$

m tangent $= -\frac{4}{3}$

Equation $y-b=m(x-a)$
$y-5=-\frac{4}{3}(x-7)$
$3y-15=-4x+28$
$3y+4x=43$.

2) Find the equations of the tangents from the point (0,-4) to the circle $x^2 + y^2 = 8$

Equation $y=mx+c$.
$y=mx-4$

Solve simultaneously
$x^2+y^2=8$
$x^2+(mx-4)^2=8$
$x^2+m^2x^2-8mx+16=8$
$x^2(1+m^2)-8mx+8=0$
$a=(1+m^2)$ $b=-8m$ $c=8$

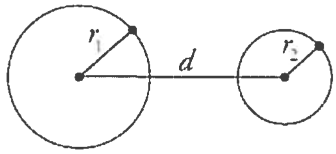
tangent ⇒ equal roots ⇒ $b^2-4ac=0$
$(-8m)^2-4(1+m^2)\times 8=0$
$64m^2-32-32m^2=0$
$32m^2=32$
$m^2=1$ ⇒ $m=\pm 1$
so $y=x-4$ or $y=-x-4$.

Ex 14C, pg 315 – 316

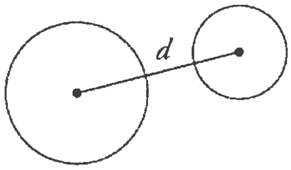
Intersecting Circles

There are five possibilities:

Consider two circles with radii r_1 and r_2 with $r_1 > r_2$.
Let d be the distance between the centres of the two circles.

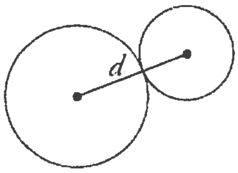


$d > r_1 + r_2$



The circles do not touch.

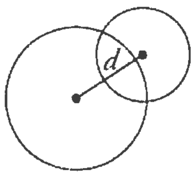
$d = r_1 + r_2$



The circles touch externally.

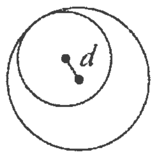
Note
Don't try to memorise this, just try to understand why each one is true.

$r_1 - r_2 < d < r_1 + r_2$



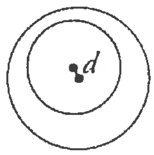
The circles meet at two distinct points.

$d = r_1 - r_2$



The circles touch internally.

$d < r_1 - r_2$



The circles do not touch.

Example
Determine how, if at all, the circles with equations $x^2 + y^2 - 8x + 6y - 11 = 0$ and $x^2 + y^2 + 4x - 10y + 4 = 0$ intersect

$x^2 + y^2 - 8x + 6y - 11 = 0$

$2g = -8 \quad 2f = 6 \quad c = -11$

$g = -4 \quad f = 3$

centre $(4, -3)$ radius $\sqrt{16 + 9 - (-11)}$

$= \sqrt{36}$

$= 6.$

$x^2+y^2+6x-10y+4=0$

$2g=4$ $\qquad$ $2f=-10$ $\qquad$ $c=4$

$g=2$ $\qquad$ $f=-5$

centre $(-2,5)$

radius $\sqrt{4+25-4}$

$=5$

distance between centres

$d=\sqrt{(4-(-2))^2+(-3-5)^2}$

$d=\sqrt{36+64}$

$d=10$

sum of radii $=5+6$

$=11$

Circles intersect in two distinct points since
distance between centres < sum of radii

Ex 14D, pg 316 – 317