Integration

The reverse process to differentiation is integration. **The integral is written**

$$\int f(x)dx$$

and is read as the indefinite integral of f(x) with respect to x.

Rules:-

- **" Increase the power by 1, then divide by the new power"**
- **Make sure you remember + c**

- $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$ (i.e. $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + c\right) = x^n$)
- $\int ax^n dx = \frac{a}{n+1} x^{n+1} + c$
- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- $\int kf(x)dx = k\int f(x)dx$ (k a constant)

c is called the constant of integration , a number which disappears on differentiation and therefore must reappear on integration.

Examples

1) $\int x^7 dx$

$= \frac{x^8}{8} + c$

2) $\int t^{-4} dt$

$= \frac{t^{-3}}{-3} + c$

$= -\frac{1}{3t^3} + c$

3) $\int 6x^3 dx$

$= \frac{6x^4}{4} + c$

$= \frac{3}{2}x^4 + c$

4) $\int (5 + 4t^{-2})dt$

$= 5t + \frac{4t^{-1}}{-1} + c$

$= 5t - \frac{4}{t} + c$

5) $\int -\frac{1}{6}x^{1/2} dx$

$= -\frac{2}{3} \times \frac{1}{6} x^{\frac{3}{2}} + c$

$= -\frac{1}{9} x^{\frac{3}{2}} + c.$

When dividing by a fraction turn it upside down and multiply by it

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AS 1.4 Applying Calculus Skills of Integration

As with differentiation the function must be in the correct form for integrating.
Deal with brackets and fractions BEFORE attempting to integrate.

6) $\int (x^2 - 3)^2 \, dx$

$= \int (x^4 - 6x^2 + 9) \, dx$

$= \frac{x^5}{5} - 2x^3 + 9x + C$

7) $\int \frac{4}{3x^4} dx$

$= \int \frac{4}{3} x^{-4} \, dx$

$= \frac{4}{3} \times \frac{x^{-3}}{-3} + C$

$= -\frac{4}{9x^3} + C$

8) $\int \frac{t + t^2 + \sqrt{t}}{t} dt$

$= \int \left(\frac{t}{t} + \frac{t^2}{t} + \frac{t^{\frac{1}{2}}}{t} \right) dt$

$= \int (1 + t + t^{-\frac{1}{2}}) \, dt$

$= t + \frac{t^2}{2} + \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= t + \frac{t^2}{2} + 2t^{\frac{1}{2}} + C$

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AS 1.4 Applying Calculus Skills of Integration

"Special" Integrals

Integrals of the form $\int (ax+b)^n \, dx$ are found as follows :-

$$\int (ax+b)^n \, dx$$

$$= \frac{1}{n+1}(ax+b)^{n+1} \cdot \frac{1}{a} + c \qquad (n = -1)$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

> **Note :-**
>
> This generalisation only applies when the contents of the bracket in the integral is linear. – i.e. in the form $ax+b$ with **no** $x^2, x^3, \sqrt{x}, \frac{1}{x}$,

Examples :-

1. $\int (8-2x)^4 \, dx$

$$= \frac{(8-2x)^5}{5 \times (-2)} + c$$

$$= -\frac{1}{10}(8-2x)^5 + c$$

2. $\int (2x+1)^{-\frac{3}{2}} \, dx$

$$= \frac{(2x+1)^{-\frac{1}{2}}}{-\frac{1}{2} \times 2} + c$$

$$= -(2x+1)^{\frac{5}{2}} + c.$$

3. $\int \frac{dx}{\sqrt{(3x+4)}}$

$$= \int (3x+4)^{-\frac{1}{2}} \, dx$$

$$= \frac{(3x+4)^{\frac{1}{2}}}{\frac{1}{2} \times 3} + c$$

$$= \frac{2}{3}(3x+4)^{\frac{1}{2}} + c$$

4. $\int 3\sqrt[5]{(4x-1)^3} \, dt$

$$= \int 3(4x-1)^{\frac{3}{5}} \, dx$$

$$= 3\,\frac{(4x-1)^{\frac{8}{5}}}{\frac{8}{5} \times 4} + c$$

$$= \frac{15}{32}(4x-1)^{\frac{8}{5}} + c.$$

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More "Special" Integrals

The above generalisation may be adapted to help integrate more complex trigonometric functions.

$\int \sin x\,dx = -\cos x + c$ $\qquad$ $\int \cos x\,dx = \sin x + c$ $\qquad$ backwards differentiation

Also $\qquad$ $\int \sin(ax+b)dx$ $\qquad$ $\int \cos(ax+b)dx$

$$= -\frac{1}{a}\cos(ax+b)+c \qquad = \frac{1}{a}\sin(ax+b)+c$$

Note :-

Again the contents of the bracket **MUST** be linear.

Examples :-

1. $\int 3\cos 4x\,dx$

$= \frac{1}{4} \cdot 3\sin 4x + c$

$= \frac{3}{4}\sin 4x + c$

2. $\int \sin\left(2x - \frac{\pi}{6}\right)dx$

$= -\frac{1}{2}\cos\left(2x - \frac{\pi}{6}\right) + c$

3. $\int -6\sin\frac{3}{2}x\ dx$

$= \frac{2}{3} \cdot 6\cos\frac{3}{2}x + c$

$= 4\cos\frac{3}{2}x + c$

4. $\int (\sin 4x + \cos(2x + \frac{\pi}{4}))dx$

$= -\frac{1}{4}\cos 4x + \frac{1}{2}\sin\left(2x + \frac{\pi}{4}\right) + c$

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Differential Equations

Equations like $\frac{dy}{dx} = x^2 - 7$ and $\frac{ds}{dt} = t^2 + 3t + 5$ are called **differential equations** (they have a derivative in them)

We solve them by **integration** to find the **general solution** (including constant) and then use additional information to find c and hence **particular solution.**

Examples

1) Write s in terms of t if $\frac{ds}{dt} = 3t^2$ and when s = 0, t = 2

$$\frac{ds}{dt} = 3t^2$$

$$s = \int 3t^2\,dt$$

$$= \frac{3t^3}{3} + c$$

remember constant of integration

$$s = t^3 + c$$

When $s = 0$, $t = 2$

use values to work out c

so $0 = 2^3 + c$

$c = -8$

so $s = t^3 - 8$

2) For every point on a curve $\frac{dy}{dx} = 3x^2 - 10x$. If the curve passes through the point (-1,0), find the equation of the curve.

$$\frac{dy}{dx} = 3x^2 - 10x$$

$$y = \int (3x^2 - 10x)\,dx$$

$$y = x^3 - 5x^2 + c$$

Point $(-1, 0)$ — x, y

so $0 = (-1)^3 - 5(-1)^2 + c$

$0 = -1 - 5 + c$

$c = 6$

$$y = x^3 - 5x^2 + 6$$

3) For a curve, it is known that f'(x) = 6 cos 3x. The point $(\frac{\pi}{2}, 3)$ lies on the curve. Find f(x).

$f'(x) = 6\cos 3x$

$f(x) = \int 6\cos 3x \, dx$

$f(x) = \frac{1}{3} \times 6\sin 3x + C.$

Goes through point $(\frac{\pi}{2}, 3)$

$\uparrow x \quad \uparrow f(x)$

$3 = \frac{1}{3} \times 6 \sin \frac{3\pi}{2} + c$

$3 = 2 \times (-1) + c$

$c = 5$

$f(x) = 2\sin 3x + 5$

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AS 1.4 Applying Calculus Skills of Integration

Definite Integrals

These are integrals where we have values for x between which we are calculating a value for the integral.

$$\int_a^b f(x)dx = F(b) - F(a) \qquad a \le x \le b$$ and F(x) is the anti-derivative of f(x).

value with upper limit − value with lower limit

NB

- Indefinite integrals give a function
- Definite integrals give a **value**
- Upper limit - lower limit

Examples

1) Evaluate $\int_1^3 x^2 dx$

definite integral → don't need +C

$= \left[\frac{x^3}{3} \right]_1^3$

after integrating use square brackets and put limits on end

$= \frac{3^3}{3} - \frac{1^3}{3}$

upper limit

lower limit

$= 9 - \frac{1}{3}$

$= 8\frac{2}{3}$

2) Evaluate $\int_{-1}^{2} t(t^2 + t^3)dt$

$= \int_{-1}^{2} (t^3 + t^4)dt$

$= \left[\frac{t^4}{4} + \frac{t^5}{5} \right]_{-1}^{2}$

$= \left(\frac{2^4}{4} + \frac{2^5}{5} \right) - \left(\frac{(-1)^4}{4} + \frac{(-1)^5}{5} \right)$

$= 4 + \frac{32}{5} - \frac{1}{4} + \frac{1}{5}$

$= 10\frac{7}{20}$

3) Evaluate $\int_{-1}^{1} (3t-1)^4 \, dt$

$= \left[\frac{(3t-1)^5}{5 \times 3} \right]_{-1}^{1}$

$= \left[\frac{1}{15}(3t-1)^5 \right]_{-1}^{1}$

$= \left(\frac{1}{15} \times 2^5 - \frac{1}{15}(-4)^5 \right)$

$= \frac{32}{15} + \frac{1024}{15}$

$= \frac{1056}{15}$

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LHC.

4) Find the positive value of z for which $\int_{1}^{z} (1+2x)dx = 4$

$\int_1^z (1+2x)\,dx = 4$

$\left[x + \frac{2x^2}{2} \right]_1^z = 4$

$\left[x + x^2 \right]_1^z = 4$

$(z+z^2) - (1+1) = 4$

$z^2 + z - 6 = 0$

$(z+3)(z-2) = 0$

$z = -3$ or $z = 2$

z is positive so $z = 2$.

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AS 1.4 Applying Calculus Skills of Integration

Definite integrals for trigonometric functions

- **remember – you need to use radians!**

1) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 3\sin x\,dx$

$$= \left[-3\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(-3\cos\frac{\pi}{2}\right) - \left(-3\cos\frac{\pi}{4}\right)$$

$$= 0 + 3\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$

2) Evaluate $\int_{0}^{\frac{\pi}{6}} 3\cos 2x\,dx$

$$= \left[\frac{1}{2} \times 3\sin 2x\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{3}{2}\sin\frac{\pi}{3}\right) - \left(\frac{3}{2}\sin 0\right)$$

$$= \frac{3}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{4}$$

3) Evaluate $\int_2^4 \frac{1}{2}\cos(3x-1)dx$ (2 s.f.)

$= \left[\frac{1}{3} \times \frac{1}{2}\sin(3x-1)\right]_2^4$

$= \left[\frac{1}{6}\sin(3x-1)\right]_2^4$

$= \frac{1}{6}\sin 11 - \frac{1}{6}\sin 5$

$= -0.0068$ (2s f)

4) Evaluate, giving your answer as a surd in its simplest form:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin x - 3\cos x dx$

$= \left[-2\cos x - 3\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left(-2\cos\frac{\pi}{3} - 3\sin\frac{\pi}{3}\right) - \left(-2\cos\frac{\pi}{6} - 3\sin\frac{\pi}{6}\right)$

$= \left(-2\times\frac{1}{2} - 3\times\frac{\sqrt{3}}{2}\right) - \left(-2\times\frac{\sqrt{3}}{2} - 3\times\frac{1}{2}\right)$

$= -1 - \frac{3\sqrt{3}}{2} + \sqrt{3} + \frac{3}{2}$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}$

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