AS 1.1 Applying Algebraic Skills To Solve Equations

Polynomials

Expressions like

$$x^4 + 3x^3 + 5x - 6$$
$$x^2 - 4$$
$$-3m^3 + 2m$$

are called **polynomials**

The **degree** of a polynomial is the value of the highest power i.e. the above polynomials are of degree 4, 2 and 3 respectively.

The numbers in front of each x^n term are called the **coefficients**, i.e. in the above the coefficient of x is 5, the coefficient of m^3 is -3, the coefficient of m^2 is zero.

Factor Theorem

$f(h) = 0 \iff (x - h)$ is a factor of $f(x)$

Examples

1) Show that $x - 1$ is a factor of $f(x) = x^3 - 3x^2 + 3x - 1$

$$f(1) = 1^3 - 3\times1^2 + 3\times1 - 1$$
$$= 1 - 3 + 3 - 1$$
$$= 0$$

$f(1) = 0$ so $x-1$ is a factor.

2) Is $x + 2$ a factor of $f(x) = x^3 - 2x^2 - x + 2$?

$$f(-2) = (-2)^3 - 2(-2)^2 - (-2) + 2$$
$$= -8 - 8 + 2 + 2$$
$$= -12$$
$$\neq 0$$

$f(-2) \neq 0$ so $x+2$ is not a factor.

Exercise :- p138 Ex 7A

$17 \div 5 = 3$ remainder 2.
so $17 = 5 \times 3 + 2$ ← remainder
divisor ↑ quotient

Factorising Polynomials

We can attempt to fully factorise polynomials using 'algebraic' long division.

Example

Given that $x + 1$ is a factor of $f(x) = x^3 + 3x^2 - 13x - 15$, factorise $f(x)$ fully.

$$
\begin{array}{r|l}
 & x^2 + 2x - 15 \\
x+1 & x^3 + 3x^2 - 13x - 15 \\
 & x^3 + x^2 \\
 & \quad 2x^2 - 13x - 15 \\
 & \quad 2x^2 + 2x \\
 & \qquad -15x - 15 \\
 & \qquad -15x - 15 \\
 & \qquad\qquad 0
\end{array}
$$

← called the quotient

← remainder

* don't use this method, next method is easier!

$f(x) = (x+1)(x^2+2x-15)$
$= (x+1)(x+5)(x-3)$

Alternatively we can use a process called 'synthetic division' to solve the same problem.

$x+1=0$
$x=-1$

← coefficients from polynomial

$$
\begin{array}{r|rrr|r}
-1 & 1 & 3 & -13 & -15 \\
 & \downarrow\text{add} & -1 & -2 & 15 \\
\hline
 & 1 \;\;{}^{\times(-1)} & 2 \;\;{}^{\times(-1)} & -15 & \boxed{0}
\end{array}
$$

← remainder 0 so $(x+1)$ is a factor

↑ gives quotient (answer after dividing)

$f(x) = (x+1)(x^2+2x-15)$
$= (x+1)(x+5)(x-3)$

In synthetic division we are calculating f(h) which is also the value of the remainder (see 'remainder theorem'), the coefficients of the quotient are also produced.

Examples

1) Show that $x - 2$ is a factor of $f(x) = x^3 - 7x + 6$ and factorise $f(x)$ fully.

$x-2=0$
$\underline{x=2}$

change sign

no x^2 term so put 0 here.

$$\begin{array}{c|cccc} 2 & 1 & 0 & -7 & 6 \\ & & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

add

$\times(2)$

remainder 0 so $(x-2)$ is a factor

$$f(x) = (x-2)(x^2+2x-3)$$
$$= (x-2)(x+3)(x-1)$$

p144 Ex 7B

2) Factorise fully $2x^3 + 5x^2 - 28x - 15$

Use trial and error.

use factors of -15 ie ±1 ±3 ±5 ±15

$$\begin{array}{c|cccc} 1 & 2 & 5 & -28 & -15 \\ & & 2 & 7 & -21 \\ \hline & 2 & 7 & -21 & \boxed{-36} \end{array}$$

$(x-1)$ is not a factor.

$$\begin{array}{c|cccc} 3 & 2 & 5 & -28 & -15 \\ & & 6 & 33 & 15 \\ \hline & 2 & 11 & 5 & \boxed{0} \end{array}$$

remainder 0 so $x-3$ is a factor

$$P(x) = (x-3)(2x^2+11x+5)$$
$$= (x-3)(x+5)(2x+1)$$

3) a) Show that $(2x-1)$ is a factor of $f(x) = 2x^3 + 5x^2 - x - 1$
   b) Hence or otherwise, fully factorise f(x)

$2x - 1 = 0$
$x = \frac{1}{2}$.

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & 5 & -1 & -1 \\ & & 1 & 3 & 1 \\ \hline & 2 & 6 & 2 & \boxed{0} \end{array}$$

remainder 0
so $x - \frac{1}{2}$ is a factor

$f(x) = (x - \frac{1}{2})(2x^2 + 6x + 2)$

$= (2x-1)(x^2 + 3x + 1)$ ie $(2x+1)$ is a factor

doesn't factorise

Exercises :- P144-150 Ex 7 B, C, D

Calculating unknown coefficients

We can use the factor theorem to calculate unknown coefficients in a polynomial.

Examples

1) If $(x+3)$ is a factor of $2x^4 + 6x^3 + px^2 + 4x - 15$

   a) Find the value of p
   b) Factorise fully the polynomial

(a) $(x+3)$ is a factor $\Rightarrow$ dividing gives remainder 0

$$\begin{array}{c|ccccc} -3 & 2 & 6 & p & 4 & -15 \\ & & -6 & 0 & -3p & -12+9p \\ \hline & 2 & 0 & p & 4-3p & -27+9p \end{array}$$

remainder 0
$\Rightarrow -27 + 9p = 0$
$p = 3$.

PTO for * space!

(b) $2x^4 + 6x^3 + 3x^2 + 4x - 15 = (x+3)(2x^3 + 3x - 5)$

(using $p=3$)

Factorise $2x^3 + 3x - 5$

$$\begin{array}{c|cccc} 1 & 2 & 0 & 3 & -5 \\ & & 2 & 2 & 5 \\ \hline & 2 & 2 & 5 & \boxed{0} \end{array}$$

remainder 0 so $(x-1)$ is a factor

$2x^4 + 6x^3 + 3x^2 + 4x - 15 = (x+3)(x-1)(2x^2 + 2x + 5)$

doesn't factorise!

2) Find the values of a and b such that $(x\text{-}2)$ and $(x\text{+}4)$ are factors of
$x^4 + ax^3 - x^2 + bx - 8$

$$
\begin{array}{r|rrrrr}
2 & 1 & a & -1 & b & -8 \\
 & & 2 & 2a+4 & 4a+6 & 8a+2b+12 \\
\hline
 & 1 & a+2 & 2a+3 & 4a+b+6 & \boxed{8a+2b+4}
\end{array}
$$

Since $(x-2)$ is a factor $\Rightarrow$ $8a+2b+4=0$

$4a+b+2=0$ ... (1)

$$
\begin{array}{r|rrrrr}
-4 & 1 & a & -1 & b & -8 \\
 & & -4 & -4a+16 & 16a-60 & -64a-4b+240 \\
\hline
 & 1 & a-4 & -4a+15 & 16a+b-60 & \boxed{-64a-4b+232}
\end{array}
$$

Since $(x+4)$ is a factor

$\Rightarrow$ $-64a-4b+232=0$

$-16a-b+58=0$ ... (2)

Solve (1) + (2) $\quad -12a+60=0$

$a=5$

In (1) $\quad 20+b+2=0$

$b=-22$

$\underline{\underline{a=5 \quad b=-22}}$

Exercise :- P151 Ex 7E

Using synthetic division we saw that on dividing $f(x)$ by $(x-h)$, $f(h)$ is the remainder.

Remainder Theorem

If a polynomial f(x) is divided by (x-h) then the remainder is f(h)

<u>Proof</u>

Consider $\quad 23 \div 5 = 4, \quad r\,3 \quad \Rightarrow \quad 23 = \underbrace{5}_{\text{divisor}} \times \underbrace{4}_{\text{quotient}} + \underbrace{3}_{\text{remainder}}$

Similarly $\quad f(x) \div (x-h) = Q(x),\ R \quad \Rightarrow \quad f(x) = \underbrace{(x-h)}_{\text{divisor}}\underbrace{Q(x)}_{\text{quotient}} + \underbrace{R}_{\text{remainder}}$

Hence for $x = h$, $f(h) = (h-h)Q(x) + R$
$= 0 \times Q(x) + R$
$= R$

Examples

1) Calculate the remainder when $4x^3 - 7x^2 + 11$ is divided by $x + 2$

$$\begin{array}{r|rrrr} -2 & 4 & -7 & 0 & 11 \\ & & -8 & 30 & -60 \\ \hline & 4 & -15 & 30 & \boxed{-49} \end{array}$$

no x term so coefficient is zero

* Watch out for missing terms!

so remainder is -49

2) Express $f(x) = 6x^4 - 4x^3 + 3x^2 + x + 8$ in the form $(x-1)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

$$\begin{array}{r|rrrrr} 1 & 6 & -4 & 3 & 1 & 8 \\ & & 6 & 2 & 5 & 6 \\ \hline & 6 & 2 & 5 & 6 & \boxed{14} \end{array}$$

quotient $6x^3 + 2x^2 + 5x + 6$ $\quad$ remainder 14.

so $f(x) = (x-1)(6x^3+2x^2+5x+6) + 14$

3) Find $(4x^4 + 2x^3 - 6x^2 + 3) \div (2x-1)$

make plain x

$= 2(x - \frac{1}{2}) \rightarrow$ use $\frac{1}{2}$ when dividing

or think $2x-1=0$
$2x=1$
$x=\frac{1}{2}$.

$$\begin{array}{c|ccccc}
\frac{1}{2} & 4 & 2 & -6 & 0 & 3 \\
 & & 2 & 2 & -2 & -1 \\
\hline
 & 4 & 4 & -4 & -2 & \boxed{2}
\end{array}$$

So $4x^4 + 2x^3 - 6x^2 + 3$

$= (x - \frac{1}{2})(4x^3 + 4x^2 - 4x - 2) + 2$.

take out common factor of 2 from second bracket into first bracket

$= (2x-1)(2x^3 + 2x^2 - 2x - 1) + 2$.

So $(4x^4 + 2x^3 - 6x^2 + 3) \div (2x-1)$ is

$\underline{2x^3 + 2x^2 - 2x - 1}$ remainder 2.

4) When $f(x) = ax^3 - 2x^2 + x - 1$ is divided by $x - 2$, the remainder is 17.
Determine the value of a.

$$\begin{array}{c|cccc}
2 & a & -2 & 1 & -1 \\
 & & 2a & 4a-4 & 8a-6 \\
\hline
 & a & 2a-2 & 4a-3 & \boxed{8a-7}
\end{array}$$

remainder is 17

so $8a - 7 = 17$

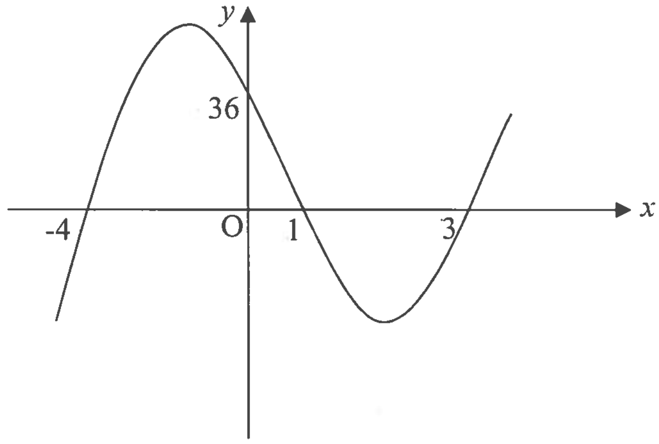
$8a = 24$

$\underline{a = 3}$.

Exercise :- P154 Ex 7F

Functions From Graphs

From the graph find an expression for f(x)



- **By considering the roots and another point on the graph, find the equation of the cubic function shown.**

Roots are $x = -4$, $x = 1$, $x = 3$

$\Rightarrow$ $x + 4 = 0$, $x - 1 = 0$ $x - 3 = 0$

so factors are $(x+4)(x-1)(x-3)$

so $f(x) = k(x+4)(x-1)(x-3)$ where k is a number to be found.

Substitute point $(0, 36)$ to find k

$36 = k(0+4)(0-1)(0-3)$

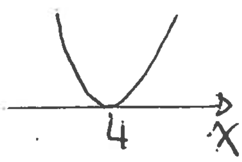
$36 = 12k$

$k = 3$

so $\underline{f(x) = 3(x+4)(x-1)(x-3)}$

Note If the curve touches the x-axis (ie x-axis is a tangent to the curve) we get a repeated root (and so a repeated factor) at that point.

eg



repeated factor $(x-4)$

ie equation contains $(x-4)^2$

Exercises :- P157 Ex 7G, p159 Ex 7H

Solving Polynomial Equations

A **root** of a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots.. + a_0$ is a value of x for which $f(x) = 0$

Recall that in solving quadratic equations we looked to factorise the quadratic, where possible.

e.g. Solve $x^2 - 2x - 8 = 0$

$(x-4)(x+2)=0$

$x=4$ or $x=-2$

The solutions are $x=4$ and $x=-2$

So $x = 4$ and $x = -2$ are the roots of the equation

In solving polynomial equations we do the same thing i.e. **factorise** the polynomial.

Example

Solve $x^3 - 4x^2 + x + 6 = 0$

Trial and error - use factors of 6 ±1 ±2 ±3 ±6

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 1 & 6 \\ & & 1 & -3 & -2 \\ \hline & 1 & -3 & -2 & \boxed{4} \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -4 & 1 & 6 \\ & & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

remainder 4 so $(x-1)$ is not a factor

remainder 0 so $(x-2)$ is a factor

$$x^3 - 4x^2 + x + 6 = (x-2)(x^2-2x-3)$$
$$= (x-2)(x-3)(x+1)$$

Solve $x^3 - 4x^2 + x + 6 = 0$

$(x-2)(x-3)(x+1)=0$

$x=2,\ x=3\ \ x=-1$

Exercise :- P163 Ex 7I

2. Solve $x^4(x+3) = 2x + 3x^2 - x^3$

$x^5 + 3x^4 = 2x + 3x^2 - x^3$

$x^5 + 3x^4 + x^3 - 3x^2 - 2x = 0$ make $= 0$

common factor

$x(x^4 + 3x^3 + x^2 - 3x - 2) = 0$

Factorise using trial and error.

$$\begin{array}{r|rrrrr} 1 & 1 & 3 & 1 & -3 & -2 \\ & & 1 & 4 & 5 & 2 \\ \hline & 1 & 4 & 5 & 2 & 0 \end{array}$$

remainder 0 so $(x-1)$ is a factor.

$x(x-1)(x^3 + 4x^2 + 5x + 2) = 0$

Factorise using trial and error again.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 5 & 2 \\ & & 1 & 5 & 10 \\ \hline & 1 & 5 & 10 & 12 \end{array}$$

$(x-1)$ is not a factor

$$\begin{array}{r|rrrr} -1 & 1 & 4 & 5 & 2 \\ & & -1 & -3 & -2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

remainder 0 so $(x+1)$ is a factor.

$x(x-1)(x+1)(x^2 + 3x + 2) = 0$

$x(x-1)(x+1)(x+2)(x+1) = 0$

$\underline{\underline{x = 0, \; x = 1, \; x = -1, \; x = -2}}$.

Exercise :- P163 Ex 7I

Finding Where a Polynomial Cuts the x and y axes

Cuts x - axis put y = 0
Cuts y - axis put x = 0

Example

$y = x^3 + 3x^2 - 13x - 15$

Find the points where the curve cuts the x and y axes.

Cuts x-axis $y=0$

$x^3+3x^2-13x-15=0$

Factorise

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -13 & -15 \\ & & 1 & 4 & -9 \\ \hline & 1 & 4 & -9 & \boxed{-24} \end{array}$$

$(x-1)$ is not a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -13 & -15 \\ & & -1 & -2 & 15 \\ \hline & 1 & 2 & -15 & \boxed{0} \end{array}$$

remainder 0
$x+1$ is a factor.

$$\begin{aligned} x^3+3x^2-13x-15 &= (x+1)(x^2+2x-15) \\ &= (x+1)(x+5)(x-3) \end{aligned}$$

cuts x-axis $(x+1)(x+5)(x-3)=0$

$x=-1,\ x=-5,\ x=3$

points $\underline{\underline{(-1,0)\ (-5,0)\ (3,0)}}$

Cuts y-axis $x=0$

$y=0+0-0-15$

$y=-15$

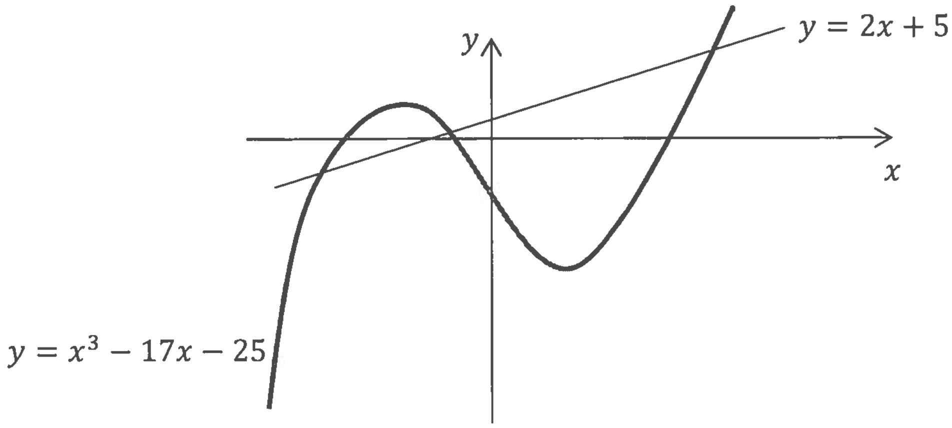
point $\underline{\underline{(0,-15)}}$

P157 Ex7G

Determining Points of Intersection on a Graphed Polynomial

Example

Determine the coordinates of the points at which the line $y = 2x + 5$ intersects the graph of $y = x^3 - 17x - 25$.



Points of intersection $\Rightarrow$ solve simultaneously

$$x^3 - 17x - 25 = 2x + 5$$
$$x^3 - 19x - 30 = 0$$

Factorise

5	1	0	−19	−30
		5	25	30
	1	5	6	0

remainder 0
so $(x-5)$ is a factor

$$x^3 - 19x - 30 = 0$$
$$(x-5)(x^2 + 5x + 6) = 0$$
$$(x-5)(x+2)(x+3) = 0$$
$$x = 5, \quad x = -2, \quad x = -3.$$

Find y co-ords using $y = 2x + 5$

$x = 5$	$x = -2$	$x = -3$
$y = 15$	$y = 1$	$y = -1$
$(5, 15)$	$(-2, 1)$	$(-3, -1)$

Exercise :- P165 Ex 7J

The Quadratic Formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

For $ax^2 + bx + c = 0$ the values of x are called the roots of the equation.

It is clear that $b^2 - 4ac$ affects what types roots we get

The Discriminant

For a quadratic equation $ax^2 + bx + c = 0$

$b^2 - 4ac$ is called the **discriminant**

If $b^2 - 4ac > 0$ then there are 2 real distinct roots (different)

note if b^2-4ac is a perfect square roots are rational

If $b^2 - 4ac = 0$ then there are equal roots (or one real root)

If $b^2 - 4ac < 0$ then there are no real roots

Note also :-

For real roots :- $b^2-4ac \geq 0$

Example

Determine the value(s) of k such that the equation $3x^2 + kx + 3 = 0$ has equal roots.

$a = 3 \quad b = k \quad c = 3.$

Equal roots $\quad b^2 - 4ac = 0$

$k^2 - 4 \times 3 \times 3 = 0$

$k^2 = 36$

$k = \pm 6$

P168 Ex 7K Q1,2

+Q3 (see notes AS 1.3 Functions ~~11~~ 13 Unit 1 p7)

Using the Discriminant

Examples

1) Determine the range of values of k for which $x^2 + kx - k = 0$ has
(a) real roots (b) no real roots

$a = 1 \quad b = k \quad c = -k$

(a) Real roots $\quad b^2 - 4ac \geq 0$

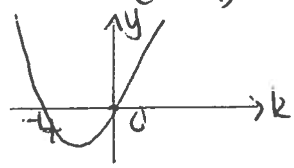
$k^2 - 4 \times 1 \times (-k) \geq 0$

$k^2 + 4k \geq 0$

Quadratic inequality => sketch graph. $y = k^2 + 4k$
$= k(k+4)$

$k^2 + 4k \geq 0$

$\Rightarrow \quad k \leq -4 \text{ or } k \geq 0$



(b) No real roots $\quad b^2 - 4ac < 0$

From graph $\quad k^2 + 4k < 0$

$-4 < k < 0$

2) Show that the roots of the equation $x^2 + kx - 2k - 4 = 0$ are always real.

$a = 1 \quad b = k \quad c = (-2k - 4)$

$b^2 - 4ac = k^2 - 4 \times 1 \times (-2k - 4)$

$= k^2 + 8k + 16$

$= (k+4)^2$ ← something squared can't be negative!

≥ 0

so roots are always real.

P168 Ex 7K Q4-10

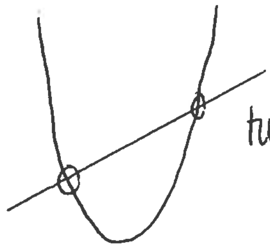
AS 1.1 Applying Algebraic Skills To Solve Equations

Intersection of a Parabola and a Straight Line

To find points of intersection solve equations simultaneously

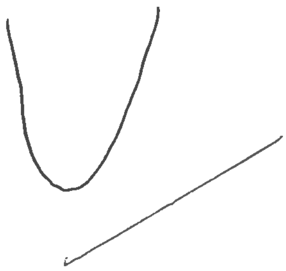
A straight line and a parabola can

(i) Intersect at two distinct points



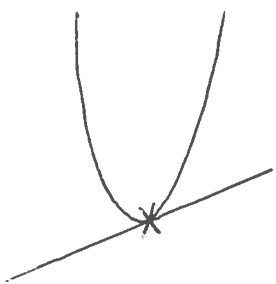
two distinct roots (solutions)

(ii) Not intersect



no solutions.
→ get quadratic eq

(iii) Intersect at one point only i.e. line is a **tangent** at that point



equal roots
↓
only one solution

Note.

* * To prove that a line is a tangent to a parabola solve their equations simultaneously and show that you get equal roots.

Examples

1) Prove that $y = 2x - 1$ is a tangent to the parabola $y = x^2$ and find the point of intersection.

Solve $y = 2x-1$
$y = x^2$

$\Rightarrow x^2 = 2x - 1$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1$

Equal roots so line is a tangent

Point of intersection $x = 1 \Rightarrow y = x^2$
$= 1$

$(1,1)$

2) If $y = mx - 9$ is the equation of a tangent to the curve with equation $y = x^2 + x - 5$, determine the value of m, given $m > 0$.

tangent $\Rightarrow$ equal roots

Solve $x^2 + x - 5 = mx - 9$

$x^2 + x - mx + 4 = 0$

$x^2 + (1-m)x + 4 = 0$ ... (*)

$b^2 - 4ac = 0$ for equal roots Here $a = 1$, $b = (1-m)$ $c = 4$

so $(1-m)^2 - 4 \times 4 = 0$

$1 - 2m + m^2 - 16 = 0$

$m^2 - 2m - 15 = 0$

$(m-5)(m+3) = 0$

$m = 5$ or $m = -3$

Since $m > 0 \Rightarrow \underline{m = 5}$

In (*) $x^2 - 4x + 4 = 0$

$(x-2)(x-2) = 0$

$x = 2$

$\Rightarrow y = 5 \times 2 - 9$
$= 1$

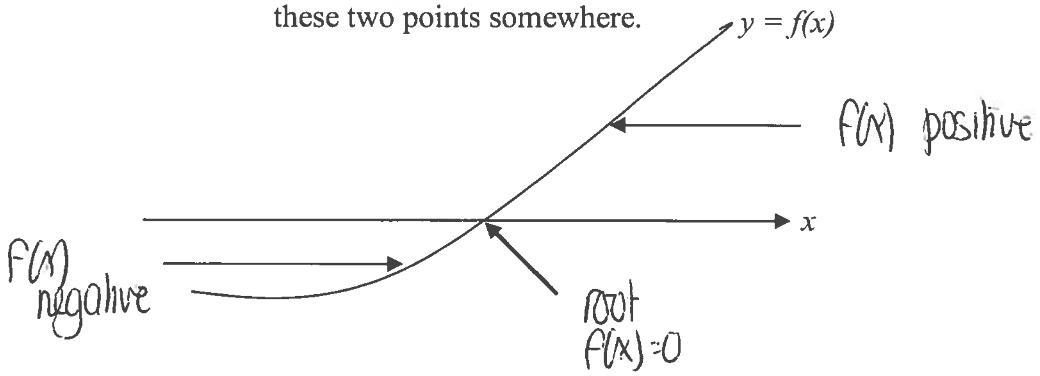
point $(2,1)$

Old book. p155 Ex8J

Approximate Roots

If a root of f(x) is not rational we can find an approximate value by a process called **iteration.**

f(x) must have a root between two values of x if f(x) is +ve for one value and –ve for the other value i.e. the graph must cross the x-axis between these two points somewhere.



We use this idea over and over again to find an approximation for the root to a given degree of accuracy.

Example

1. Show that f(x) = x^3 -4x^2 -2x +7 has a real root between 1 and 2 and find this root to decimal places.

Find f(1) and f(2) and see if there is a change of sign.

$f(1) = 1-4-2+7 = 2$ positive
$f(2) = 8-16-4+7 = -5$ negative
} change of sign so root lies between 1 and 2.

* Use calculator mode 3 - table.

f(x) = ^ use alpha x (in red)

= start 1

= end 2

= step 0.1 repeat.

$f(1.2) = 0.568$ $f(1.3) = -0.163$ ⇒ root between 1.2 and 1.3

$\boxed{AC}$ to Repeat = start 1.2 end 1.3 step 0.01

$f(1.27) = 0.0567$ $f(1.28) = -0.016$ ⇒ root between 1.27 and 1.28

Repeat = start 1.27 end 1.28 step 0.001

$f(1.277) = 5.5 \times 10^{-3}$ $f(1.278) = -1 \times 10^{-3}$ ⇒ root between 1.277 and 1.278.

Root is 1.28 (to 2d.p)

2. The line $y = 2x + 2$ intersects the curve $y = x^3 - 5x^2 - 3x + 1$ between $x=5$ and $x=6$. Find the x-coordinate (to 2 d.p.) of this point of intersection.

Point of intersection $\Rightarrow$ solve simultaneously

$$x^3 - 5x^2 - 3x + 1 = 2x + 2$$

$$x^3 - 5x^2 - 5x - 1 = 0$$

Let $f(x) = x^3 - 5x^2 - 5x - 1$

$\left.\begin{array}{l} f(5) = -26 \\ f(6) = 5 \end{array}\right\}$ change of sign so root is between $x=5$ and $x=6$

$\left.\begin{array}{l} f(5.8) = -3.088 \\ f(5.9) = 0.829 \end{array}\right\}$ root between 5.8 and 5.9

$\left.\begin{array}{l} f(5.87) = -0.372 \\ f(5.88) = 0.0254 \end{array}\right\}$ change of sign root between 5.87 and 5.88

$\left.\begin{array}{l} f(5.879) = -0.014 \\ f(5.880) = 0.0254 \end{array}\right\}$ change of sign

root is $\underline{5.88}$ (to 2d.p)

Old book.

Exercise:- HHM Pg 138 Ex 7J