AS 1.4 Applying Geometric Skills To Vectors

You should be familiar with the vectors topic which was covered at National 5 level.

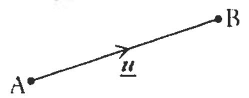
Vectors & Scalars

A **scalar** quantity has **magnitude** (size) only.

A **vector** quantity has both **magnitude** and **direction**.

A vector can be represented geometrically by a directed line segment.

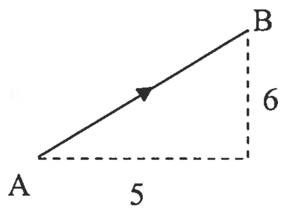
This is the vector $\overrightarrow{AB}$ or $\underline{u}$.



Components

A vector may be represented by its components. These define how to get from one end of the vector to the other. For example:

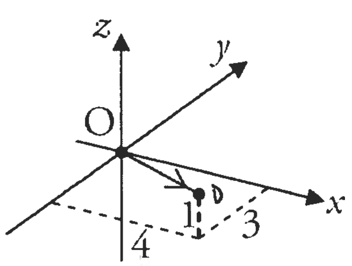
In Two Dimensions



In component form

$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ ← 5 along (x direction)
← 6 up (y direction)

In Three Dimensions



In component form.

$\overrightarrow{OD} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

Magnitude

The **magnitude** (or length) of a vector $\underline{u}$ is written as $|\underline{u}|$ or $|\overrightarrow{AB}|$.
It can be calculated as follows:

If $\overline{PQ} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $|\overline{PQ}| = \sqrt{a^2 + b^2}$.

If $\overline{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $|\overline{PQ}| = \sqrt{a^2 + b^2 + c^2}$.

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{u}|$.

$$|\underline{u}| = \sqrt{5^2 + (-12)^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13$$

2. Find the length of $\underline{a} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

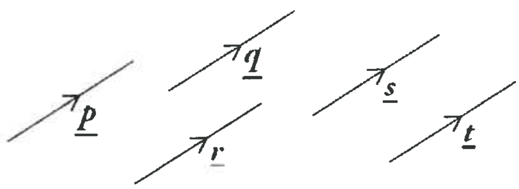
$$|\underline{a}| = \sqrt{(-\sqrt{5})^2 + 6^2 + 3^2}$$
$$= \sqrt{5 + 36 + 9}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

Equal Vectors

Vectors with the same magnitude and direction are said to be equal.

For example, all the vectors shown to the right are equal.

If vectors are equal to each other, then all of their components are equal, i.e.



if $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ then $a = d$, $b = e$ and $c = f$.

Conversely, two vectors are only equal if all of their components are equal.

AS 1.4 Applying Geometric Skills To Vectors

Negative Vectors

The negative of a vector is the vector multiplied by -1.

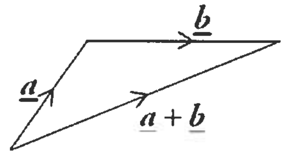
If we write a vector as a directed line segment $\overrightarrow{AB}$, then $-\overrightarrow{AB} = \overrightarrow{BA}$:

Note $\underline{u} + - \underline{u} = \mathbf{0}$

The negative of $\underline{u} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is $-\underline{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

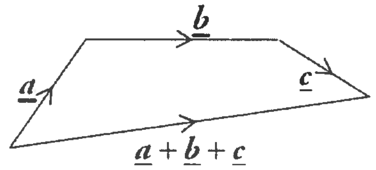
Addition & Subtraction of Vectors

We can construct $\underline{a} + \underline{b}$ as follows:



$\underline{a} + \underline{b}$ means $\underline{a}$ followed by $\underline{b}$.

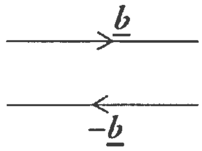
Similarly, we can construct $\underline{a} + \underline{b} + \underline{c}$ as follows:



$\underline{a} + \underline{b} + \underline{c}$ means $\underline{a}$ followed by $\underline{b}$ followed by $\underline{c}$.

To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

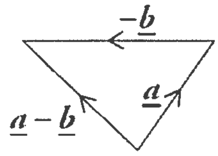
Now consider $\underline{a}-\underline{b}$. This can be written as $\underline{a}+(-\underline{b})$, so if we first find $-\underline{b}$ we can use vector addition to obtain $\underline{a}-\underline{b}$.



$-\underline{b}$ is just $\underline{b}$ but in the opposite direction.

$-\underline{b}$ and $\underline{b}$ have the same magnitude, i.e. $|\underline{b}|=|-\underline{b}|$.

Therefore we can construct $\underline{a}-\underline{b}$ as follows:



$\underline{a}-\underline{b}$ means $\underline{a}$ followed by $-\underline{b}$.

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}+\begin{pmatrix} d \\ e \\ f \end{pmatrix}=\begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

add the components

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}-\begin{pmatrix} d \\ e \\ f \end{pmatrix}=\begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

subtract the components

EXAMPLES

1. Given $\underline{u}=\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{v}=\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{u}+\underline{v}$ and $\underline{u}-\underline{v}$.

$$\underline{u}+\underline{v}=\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}+\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}$$

$$\underline{u}-\underline{v}=\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}-\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

2. Given $\underline{p}=\begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ and $\underline{q}=\begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{p}-\underline{q}$ and $\underline{q}+\underline{p}$.

$$\underline{p}-\underline{q}=\begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}-\begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}=\begin{pmatrix} 5 \\ -\frac{3}{2} \\ \frac{21}{5} \end{pmatrix}$$

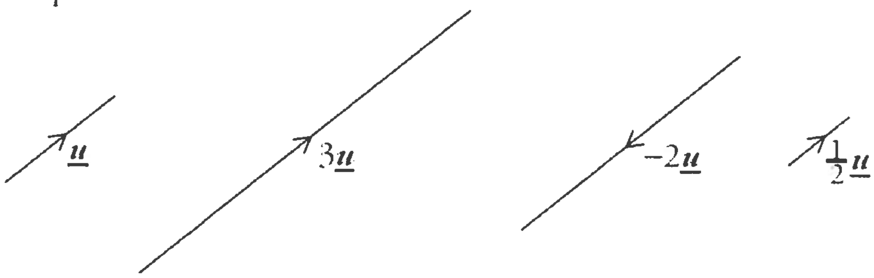
$$\underline{q}+\underline{p}=\begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}+\begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}=\begin{pmatrix} 3 \\ \frac{9}{2} \\ \frac{9}{5} \end{pmatrix}$$

Multiplication by a Scalar

A vector $\underline{u}$ which is multiplied by a scalar $k>0$ will give the result $k\underline{u}$. This vector will be k times as long, i.e. its magnitude will be $k|\underline{u}|$.

Note that if $k<0$ this means that the vector $k\underline{u}$ will be in the opposite direction to $\underline{u}$.

For example:



$$\text{If} \quad \underline{u}=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad k\underline{u}=\begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}.$$

Each component is multiplied by the scalar.

Examples

1. Three vectors are defined by $\underline{u}=\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\underline{v}=\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\underline{w}=\begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$

Find the magnitude of $\underline{u}-\underline{v}+2\underline{w}$.

$$\underline{u}-\underline{v}+2\underline{w} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -8 \\ 10 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 9 \\ 5 \end{pmatrix}$$

$$|\underline{u}-\underline{v}+2\underline{w}| = \sqrt{(-10)^2+9^2+5^2} = \sqrt{206}$$

2. Given $\underline{u}=\begin{pmatrix} 3x \\ 5+y \\ 9 \end{pmatrix}$ and $\underline{v}=\begin{pmatrix} 12 \\ 8 \\ Z-2 \end{pmatrix}$ and $\underline{u}=\underline{v}$, calculate x, y and z.

$$\underline{u}=\underline{v} \qquad \begin{pmatrix} 3x \\ 5+y \\ 9 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ z-2 \end{pmatrix}$$

so $3x=12$, $x=4$ $\qquad$ $5+y=8$, $y=3$ $\qquad$ $9=z-2$, $z=11$

Unit Vectors

Any vector with a magnitude of one is called a **unit vector**. For example:

if $\underline{u} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ then $|\underline{u}| = \sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= 1$

So $\underline{u}$ is a unit vector.

As the magnitude of vector $\underline{v}$ is $|\underline{v}|$, the unit vector in the direction of $\underline{v}$ is $\frac{1}{|\underline{v}|}\underline{v}$

Example

If $\underline{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find the unit vector in the direction of $\underline{a}$.

$|\underline{a}| = \sqrt{9+16}$

$= 5$

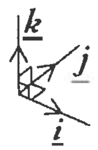
unit vector $\underline{u} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$

p98 EX5A
1 5 7 8 9
10 → 13

Basis Vectors - *(Special Unit Vectors)*

A vector may also be defined in terms of the basis vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.

These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).



These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using $\underline{i}$, $\underline{j}$ and $\underline{k}$. For example:

$$\underline{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}.$$

There is no need for the working above if the following is used:

$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

AS 1.4 Applying Geometric Skills To Vectors

Examples

1. Express the following in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

a) $\underset{\sim}{p} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ b) $\underset{\sim}{q} = \begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix}$ c) $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$

(a) $\underset{\sim}{p} = 5\underset{\sim}{i} - 3\underset{\sim}{j}$

(b) $\underset{\sim}{q} = -4\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{k}$

2. If $\underset{\sim}{p} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{q} = \underset{\sim}{i} - 3\underset{\sim}{j}$

a) Calculate $3\underset{\sim}{p} - 2\underset{\sim}{q}$ giving your answer in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

b) Calculate the magnitude of $3\underset{\sim}{p} - 2\underset{\sim}{q}$ leaving your answer as a surd.

(a) $3\underset{\sim}{p} - 2\underset{\sim}{q} = 3(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) - 2(\underset{\sim}{i} - 3\underset{\sim}{j})$

$= 6\underset{\sim}{i} - 3\underset{\sim}{j} + 9\underset{\sim}{k} - 2\underset{\sim}{i} + 6\underset{\sim}{j}$

$= 4\underset{\sim}{i} + 3\underset{\sim}{j} + 9\underset{\sim}{k}$

(b) $|3\underset{\sim}{p} - 2\underset{\sim}{q}| = \sqrt{4^2 + 3^2 + 9^2}$

$= \sqrt{106}$

Ex 5A · 2 3 4 6 14

+ complete any other unfinished Q

Ex 5A, pg 98 - 99

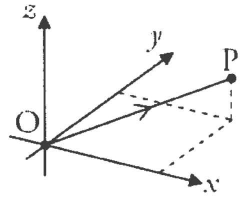
AS 1.4 Applying Geometric Skills To Vectors

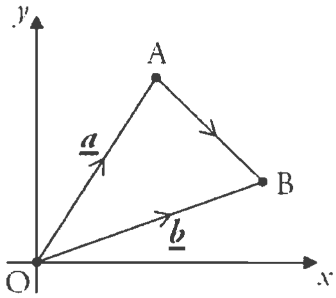
Position Vectors

$\overline{OA}$ is called the **position vector** of point A relative to the origin O, and is written as $\underline{a}$.

$\overline{OB}$ is called the position vector of point B, written $\underline{b}$.

Given $P(x, y, z)$, the position vector $\overline{OP}$ or $\underline{p}$ has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.





To move from point A to point B we can move back along the vector $\underline{a}$ to the origin, and along vector $\underline{b}$ to point B, i.e.

$$\begin{aligned}\overline{AB} &= \overline{AO} + \overline{OB} \\ &= -\overline{OA} + \overline{OB} \\ &= -\underline{a} + \underline{b} \\ &= \underline{b} - \underline{a}.\end{aligned}$$

For the vector joining any two points P and Q, $\overline{PQ} = \underline{q} - \underline{p}$.

Examples

1. R is the point (2, -2, 3) and S is the point (4, 6, -1). Find $\overrightarrow{RS}$.

$$\overrightarrow{RS} = \underline{s} - \underline{r} = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}$$

2. If A is the point (3, 7) and the vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, calculate the coordinates of B.

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\underline{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \underline{a}$$
$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

B(5,8)

remember to turn vector 'on side' to get point

3. *ABCD* is a parallelogram. A is the point (2, -1, 4), B(7, 1, 3) and C(-6, 4, 2).

Calculate the coordinates of D.

$$\overrightarrow{CD} = \overrightarrow{BA}$$
$$\underline{d} - \underline{c} = \underline{a} - \underline{b}$$
$$\underline{d} = \underline{a} - \underline{b} + \underline{c}$$

$$\underline{d} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -6 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 2 \\ 3 \end{pmatrix}$$

D(-11,2,3)

Ex 5B, pg 102

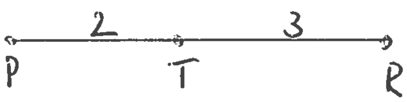
Dividing Lines In A Ratio

There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

EXAMPLE

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

   The point T divides PR in the ratio $2:3$. Find the coordinates of T.



$$\frac{PT}{TR} = \frac{2}{3}$$

$$3PT = 2TR.$$

$$\text{so } 3\overrightarrow{PT} = 2\overrightarrow{TR}$$

$$3(\underline{t} - \underline{p}) = 2(\underline{r} - \underline{t})$$

$$3\underline{t} - 3\underline{p} = 2\underline{r} - 2\underline{t}$$

$$5\underline{t} = 2\underline{r} + 3\underline{p}$$

$$\underline{t} = \frac{1}{5}(2\underline{r} + 3\underline{p})$$

Make t the subject → this is what we want to find

$$\underline{t} = \frac{1}{5}\left(2\begin{pmatrix}8\\-1\\19\end{pmatrix} + 3\begin{pmatrix}-2\\4\\-1\end{pmatrix}\right)$$

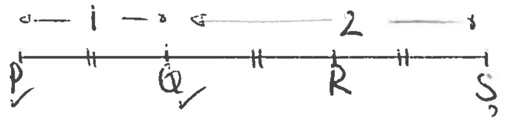
$$= \frac{1}{5}\left(\begin{pmatrix}16\\-2\\38\end{pmatrix} + \begin{pmatrix}-6\\12\\-3\end{pmatrix}\right)$$

$$= \frac{1}{5}\begin{pmatrix}10\\10\\35\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}2\\2\\7\end{pmatrix} \qquad T(2, 2, 7)$$

↑ vector ↑ point

2. PQRS is a straight line. P and Q are the points (-1, 3, 2) and (5, -1, 3).
   If PQ = QR = RS find the coordinates of S.



$$\frac{PQ}{QS} = \frac{1}{2}$$

$$2PQ = QS$$

$$\text{so } 2\overrightarrow{PQ} = \overrightarrow{QS}$$

$$2(\underline{q} - \underline{p}) = \underline{s} - \underline{q}$$

$$2\underline{q} - 2\underline{p} = \underline{s} - \underline{q}$$

$$\underline{s} = 3\underline{q} - 2\underline{p}$$

$$\underline{s} = 3\begin{pmatrix}5\\-1\\3\end{pmatrix} - 2\begin{pmatrix}-1\\3\\2\end{pmatrix}$$

$$\underline{s} = \begin{pmatrix}15\\-3\\9\end{pmatrix} - \begin{pmatrix}-2\\6\\4\end{pmatrix} = \begin{pmatrix}17\\-9\\5\end{pmatrix}$$

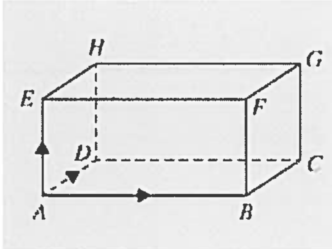
so $S = (17, -9, 5)$

Ex 5C, pg 104-105

Determining the resultant of vector pathways in 3-dimensions

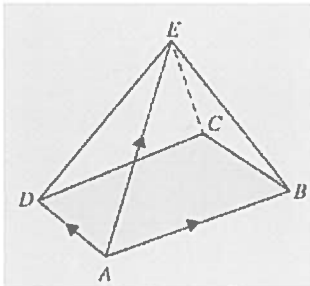
Examples

1. *ABCDEFGH* is a cuboid.
Find a single vector equivalent to:
$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$
This question could also be written as:
Find the resultant of $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$.



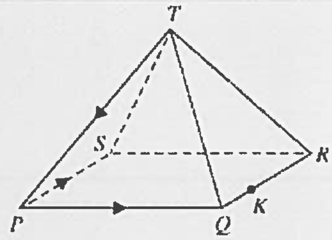
$$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$$
$$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG}$$
$$= \overrightarrow{AG}$$

2. *ABCDE* is a pyramid with a rectangular base *ABCD*.
$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -1 \\ 9 \\ -2 \end{pmatrix}$, $\overrightarrow{AE} = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$
Express the vector $\overrightarrow{CE}$ in component form.



$$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DA} + \overrightarrow{AE}$$
$$= -\overrightarrow{AB} - \overrightarrow{AD} + \overrightarrow{AE}$$
$$= -\begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 9 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} -6 \\ -6 \\ 6 \end{pmatrix}$$

3. *PQRST* is a pyramid with a rectangular base.
K divides *QR* in the ratio 1:2.
$\overrightarrow{TP}$ represents $-3\underset{\sim}{i} - 5\underset{\sim}{j} - 6\underset{\sim}{k}$
$\overrightarrow{PQ}$ represents $5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$
$\overrightarrow{PS}$ represents $6\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$
Find the vector $\overrightarrow{TK}$ in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.



$$\overrightarrow{TK} = \overrightarrow{TP} + \overrightarrow{PQ} + \overrightarrow{QK}$$
$$= \overrightarrow{TP} + \overrightarrow{PQ} + \tfrac{1}{3}\overrightarrow{QR}$$
$$= \overrightarrow{TP} + \overrightarrow{PQ} + \tfrac{1}{3}\overrightarrow{PS}$$
$$= -3\underline{i} - 5\underline{j} - 6\underline{k} + 5\underline{i} + \underline{j} - 2\underline{k} + \tfrac{1}{3}(6\underline{i} - 3\underline{j} + 3\underline{k})$$
$$= 2\underline{i} - 4\underline{j} - 8\underline{k} + 2\underline{i} - \underline{j} + \underline{k}$$
$$= 4\underline{i} - 5\underline{j} - 7\underline{k}$$

Ex 5D, pg 107 - 109

<u>Parallel Vectores & Collinearity</u>

Points are collinear if they lie on the same straight line.

> The points A, B and C in 3D space are collinear if $\overline{AB}$ is parallel to $\overline{BC}$, with B a common point.

Note that we *cannot* find gradients in three dimensions – instead we use the following.

> Non-zero vectors are **parallel** if they are scalar multiples of the same vector.

For example:

$$\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underline{u}.$$

So $\underline{u}$ and $\underline{v}$ are parallel.

$$\underline{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}, \quad \underline{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

So $\underline{p}$ and $\underline{q}$ are parallel.

Examples

1. A is the point $(1, -2, 5)$, $B(8, -5, 9)$ and $C(22, -11, 17)$.
   Show that A, B and C are collinear.

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}$$

$$\vec{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 22 \\ -11 \\ 17 \end{pmatrix} - \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} = \begin{pmatrix} 14 \\ -6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}$$

so $\quad 2\vec{AB} = \vec{BC}$

so AB and BC are parallel and since they have a point in common A, B, and C are collinear.

2. Prove that the points A(2, -3, 4) , B(12, 7, -1) and C(8, 3, 1) are collinear and determine the ratio in which C divides AB.

Find vectors $\vec{AC}$ and $\vec{CB}$

$$\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\vec{CB} = \underline{b} - \underline{c} = \begin{pmatrix} 12 \\ 7 \\ -1 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

make sure numbers are correct way round!

so $\quad 2\vec{AC} = 3\vec{CB}$

so AC and CB are parallel.
and since they have a point in common A, B and C are collinear

$$2AC = 3CB$$

$$\frac{AC}{CB} = \frac{3}{2} \qquad \text{so} \qquad AC : CB = 3 : 2$$

2. If the vectors $\underline{u} = \begin{pmatrix} a \\ 6 \\ c \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ are parallel, calculate the values of a and c.

parallel so $\underline{u} = k\underline{v}$ for some $k \in R$

$$\begin{pmatrix} a \\ 6 \\ c \end{pmatrix} = k\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

$a = 3k \qquad 6 = 2k \qquad c = 4k$

$k = 3$

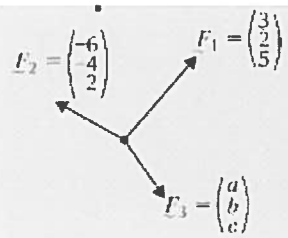
so $\underline{\underline{a = 9}}$ and $\underline{\underline{c = 12}}$

Ex 5E, pg 112-113

Zero Vectors

Any vector with all its components zero is called a zero vector and can be written as $\underline{0}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$.

Example



Three forces, measured in newtons, are acting on a point as shown in the diagram.

If the point is stationary, calculate *a*, *b* and *c*.

Stationary $\Rightarrow$ no overall force.

ie $\underline{F_1} + \underline{F_2} + \underline{F_3} = \underline{0}$

$$\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{0}$$

$$\begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{0}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}$$

$a = 3 \quad b = 2 \quad c = -7$

Ex 6A, pg 115 - 116

Scalar Product

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.

The **scalar product** is denoted by $\underline{a}.\underline{b}$ (sometimes it is called the dot product) and can be calculated using the formula:

$$\boxed{\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta}$$

where θ is the angle between the two vectors $\underline{a}$ and $\underline{b}$.

- The definition above assumes that the vectors ***a*** and ***b*** are positioned so that they both point away from the angle.

- The scalar product gives a number answer, not a vector.

- for $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\quad \underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$

- $\underline{a}.\underline{a} = |a|^2$

- $\cos\theta =$

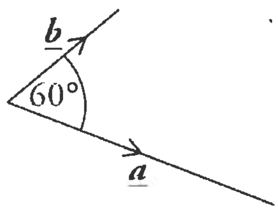
- $\underline{a}.\underline{b} > 0$, θ acute

- $\underline{a}.\underline{b} < 0$, θ obtuse

- $\underline{a}.\underline{b} = 0$, θ is 90° (vectors are perpendicular)

EXAMPLES

1. Two vectors, $\underline{a}$ and $\underline{b}$ have magnitudes 7 and 3 units respectively and are at an angle of $60°$ to each other as shown below.



What is the value of $\underline{a}.\underline{b}$?

$\underline{a} . \underline{b} = |\underline{a}||\underline{b}| \cos\theta$
$= 7 \times 3 \times \cos 60$
$= 7 \times 3 \times \frac{1}{2}$
$= 10.5$

2. Find $\underline{p}.\underline{q}$, given that $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

$\underline{p} . \underline{q} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$
$= 2 + 4 - 9$
$= -3$

3. If A is the point $(2, 3, 9)$, $B(1, 4, -2)$ and $C(-1, 3, -6)$, calculate $\overrightarrow{AB}.\overrightarrow{AC}$.

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$
$= \begin{pmatrix} -1 \\ 1 \\ -11 \end{pmatrix}$

$\overrightarrow{AC} = \underline{c} - \underline{a}$
$= \begin{pmatrix} -1 \\ 3 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$
$= \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix}$

$\overrightarrow{AB} . \overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ -11 \end{pmatrix} . \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix}$
$= 3 + 0 + 165$
$= 168$

Ex 6B, pg 119 – 120

The Angles Between Vectors

The formulae for the scalar product can be rearranged to give the following equations, both of which can be used to calculate θ, the angle between two vectors.

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} \qquad \text{or} \qquad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}.$$

Look back to the formulae for finding the scalar product, given on the previous pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply substitutes $\underline{a}.\underline{b}$ for the component form of the scalar product.

These formulae are *not* given in the exam but can both be easily derived from the formulae on the previous pages (which *are* given in the exam).

EXAMPLES

1. Calculate the angle θ between vectors $\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ and $\underline{q} = 4\underline{i} + \underline{j} + 3\underline{k}$.

$$\underline{p} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \qquad \underline{q} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$|\underline{p}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \sqrt{29}$$

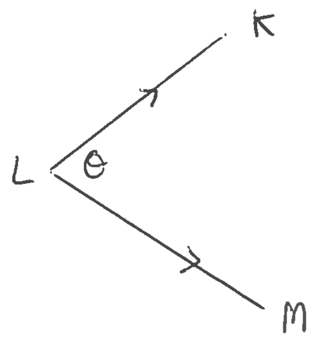
$$|\underline{q}| = \sqrt{4^2 + 1^2 + 3^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$$

$$\underline{p}\cdot\underline{q} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = 12 + 4 - 6 = 10$$

$$\cos\theta = \frac{\underline{p}\cdot\underline{q}}{|\underline{p}||\underline{q}|} = \frac{10}{\sqrt{29}\sqrt{26}} = 0.364\ldots$$

$$\theta = 68.6^\circ \quad (1\,\text{d.p.})$$

2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find $K\hat{L}M$.



Draw a sketch to make sure you use vectors which point outwards.

need $\overrightarrow{LK}$ and $\overrightarrow{LM}$

$$\overrightarrow{LK} = \underline{k} - \underline{l} = \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix}$$

$$\overrightarrow{LM} = \underline{m} - \underline{l} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{LK} = \sqrt{4^2 + (-10)^2 + (-2)^2} = \sqrt{16 + 100 + 4} = \sqrt{120}$$

$$\overrightarrow{LM} = \sqrt{5^2 + 2^2 + (-3)^2} = \sqrt{25 + 4 + 9} = \sqrt{38}$$

$$\overrightarrow{LK} . \overrightarrow{LM} = \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix} . \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} = 20 - 20 + 6 = 6$$

$$\cos\theta = \frac{\overrightarrow{LK} . \overrightarrow{LM}}{|\overrightarrow{LK}|\,|\overrightarrow{LM}|} = \frac{6}{\sqrt{120}\sqrt{38}}$$

$$\underline{\underline{\theta = 84.9^\circ}}$$

Ex 6C, pg 123 – 125

Perpendicular Vectors

If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a}.\underline{b} = 0$.

This is because $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$= |\underline{a}||\underline{b}|\cos 90°$ ($\theta = 90°$ since perpendicular)

$= 0$ (since $\cos 90° = 0$).

Conversely, if $\underline{a}.\underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendicular.

EXAMPLES

1. Two vectors are defined as $\underline{a} = 4\underline{i} + 2\underline{j} - 5\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$.

   Show that $\underline{a}$ and $\underline{b}$ are perpendicular.

$$\underline{a} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\underline{a}.\underline{b} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$= 8 + 2 - 10$$

$$= 0$$

so $\underline{a}$ and $\underline{b}$ are perpendicular.

2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where a is a constant.

   Given that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular, find the value of a.

$\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular

$$\Rightarrow \overrightarrow{PQ} . \overrightarrow{RS} = 0$$

$$\begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix} = 0$$

$$8 - 3a + 7a = 0$$

$$4a = -8$$

$$a = -2.$$

Ex 6D, pg 127 – 128

Properties of the Scalar Product

Some useful properties of the scalar product are as follows:

$$\underline{a}.\underline{b} = \underline{b}.\underline{a}$$

$$\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c} \qquad \text{(Expanding brackets)}$$

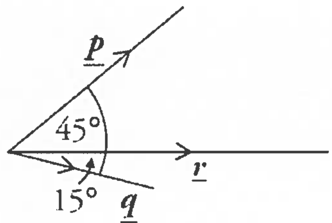
$$\underline{a}.\underline{a} = |\underline{a}|^2.$$

Note that these are not given in the exam, so you need to remember them.

EXAMPLES

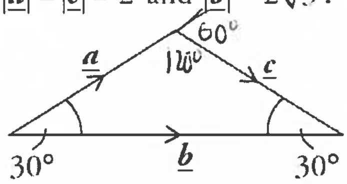
1. In the diagram, $|\underline{p}| = 3$, $|\underline{r}| = 4$ and $|\underline{q}| = 2$.

   Calculate $\underline{p}.(\underline{q}+\underline{r})$.



$$\underline{p}.(\underline{q}+\underline{r})$$
$$= \underline{p}.\underline{q} + \underline{p}.\underline{r}$$
$$= |\underline{p}||\underline{q}|\cos 60 + |\underline{p}||\underline{r}|\cos 45$$
$$= 3\times 2\times \tfrac{1}{2} + 3\times 4\times \tfrac{1}{\sqrt{2}}$$
$$= 3 + \frac{12}{\sqrt{2}}$$
$$= 3 + 6\sqrt{2}$$

2. In the diagram below, $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.



   Calculate $\underline{a}.(\underline{a}+\underline{b}+\underline{c})$.

$$\underline{a}.(\underline{a}+\underline{b}+\underline{c}) = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{a}.\underline{c}$$
$$= |\underline{a}|^2 + |\underline{a}||\underline{b}|\cos 30 + |\underline{a}||\underline{c}|\cos 60$$
$$= 2^2 + 2\times 2\sqrt{3}\times \frac{\sqrt{3}}{2} + 2\times 2\times \frac{1}{2}$$
$$= 4+6+2$$
$$= 12.$$

Ex 6E, pg 129 – 130