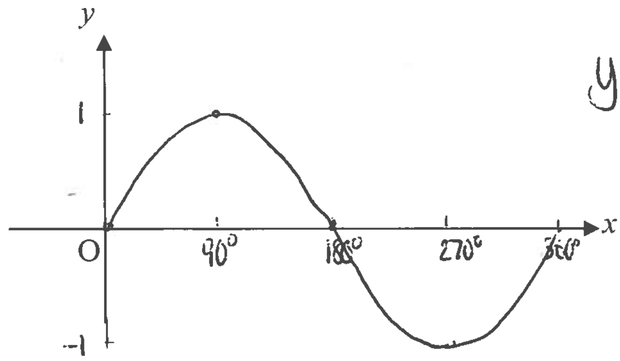
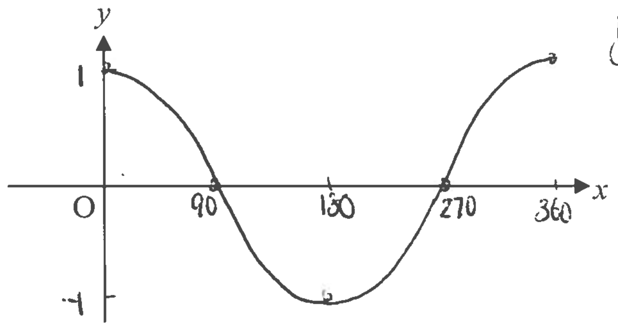
Trigonometric Graphs



$y = \sin x$

amplitude 1
period 360°

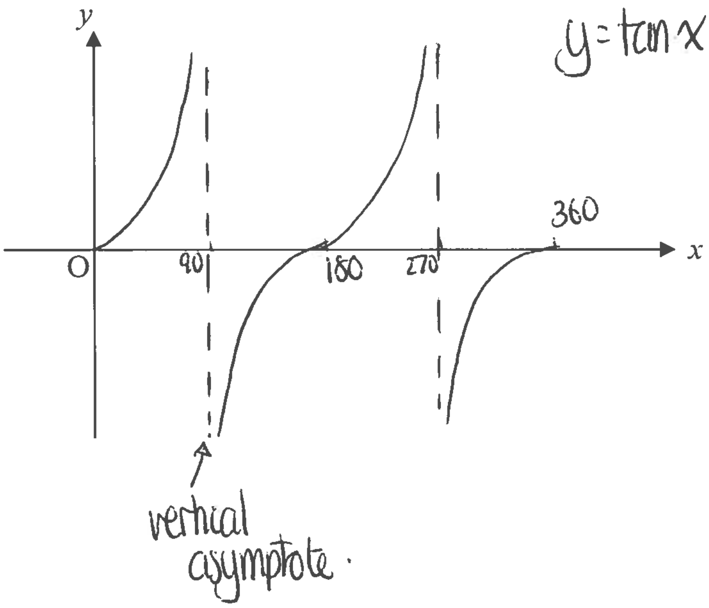
maximum 1 minimum -1



$y = \cos x$

amplitude 1
period 360°

maximum 1 minimum -1

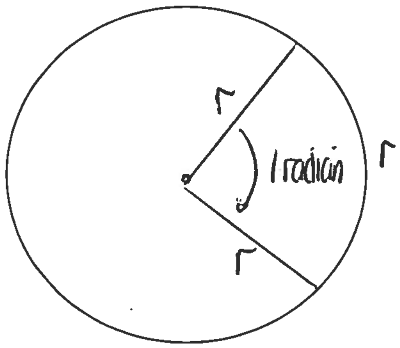


$y = \tan x$

amplitude undefined
period 180°

Radians

Radians are another unit of measurement for angles.
The angle subtended at the centre of a circle by an arc equal in length to the radius is **1 radian**.



The radius of the circle will fit on the circumference 2 π times.
So there are 2 π radians in a complete turn.
Hence 2 π radians = 360°

$C = \pi d$
$C = 2\pi r$

$$\boxed{\pi \textbf{ radians} = \textbf{180}°}$$

Also we can see that $90° = \dfrac{\pi}{2}$

$45° = \dfrac{\pi}{4}$

$60° = \dfrac{\pi}{3}$

$30° = \dfrac{\pi}{6}$

$1° = \dfrac{\pi}{180}$

It follows that we can now convert to and from radians.

Examples

1) Convert 135° to radians

$135° \Rightarrow 3 \times 45° \Rightarrow \dfrac{3\pi}{4}$ radians

or $135° \Rightarrow 135 \times \dfrac{\pi}{180} = \dfrac{3\pi}{4}$ radians.

2) Convert 120° to radians

$$120^\circ = 2 \times 60^\circ \Rightarrow \frac{2\pi}{3} \text{ radians.}$$

$$\text{or } \quad 120^\circ \Rightarrow \frac{120}{180}\pi = \frac{2}{3}\pi \text{ radians}$$

3) Convert $\frac{\pi}{5}$ to degrees

$$\frac{\pi}{5} \Rightarrow \frac{180^\circ}{5} = 36^\circ$$

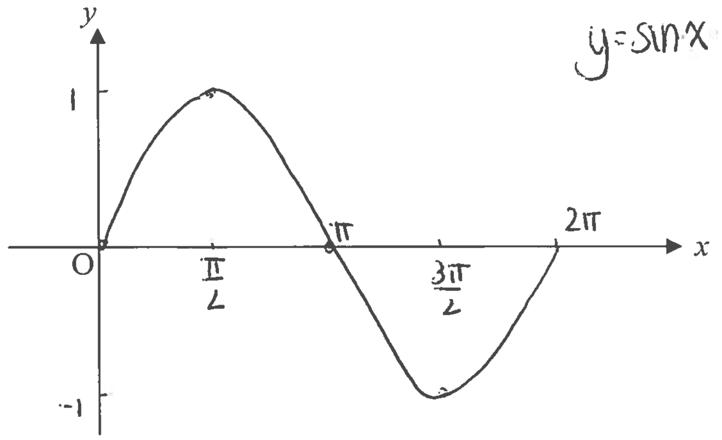
(since $\pi = 180^\circ$ radians)

4) Convert $\frac{5\pi}{6}$ to degrees

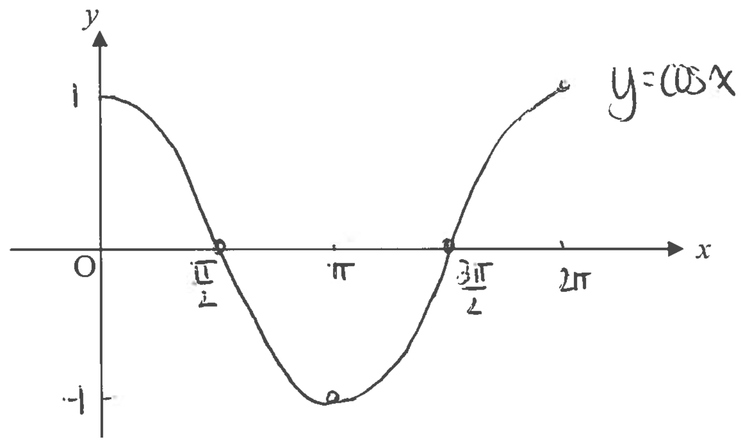
$$\frac{5\pi}{6} \Rightarrow \frac{5 \times 180^\circ}{6} = 150^\circ$$

p31 Ex 2B Q1-3

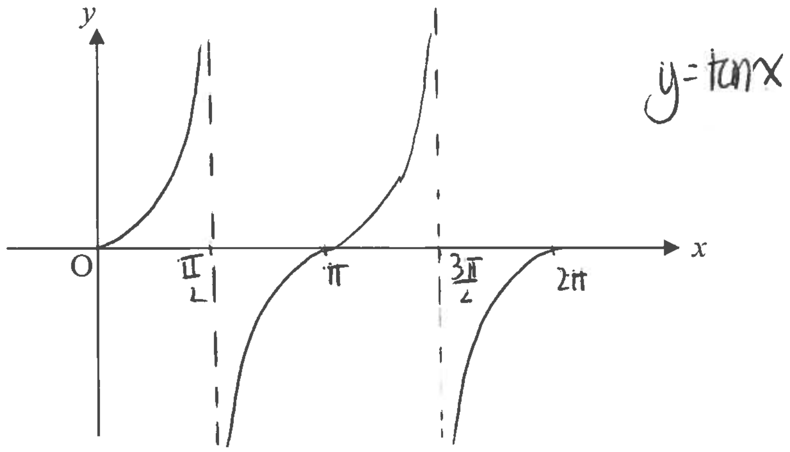
Trigonometric graphs with radians



amplitude 1
period 2π



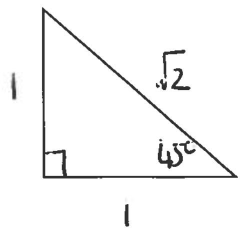
amplitude 1
period 2π

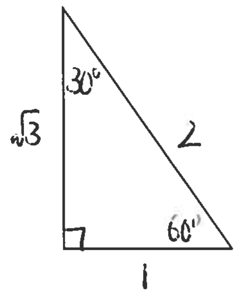


amplitude undefined
period π

Exact Value Triangles

Recall

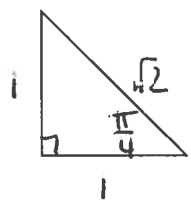


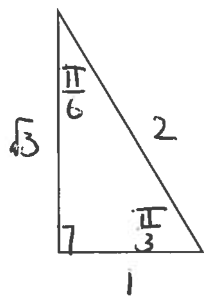


This can also be shown in a table:-

x°	**30°**	**45°**	**60°**
sinx°	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cosx°	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tanx°	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Obviously these could be written with radians





x	$\pi/6$	$\pi/4$	$\pi/3$
sinx	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cosx	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tanx	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

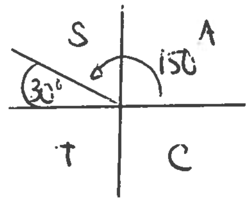
Examples

1) Find the exact values of :- a) $\cos 150^\circ$ b) $\sin 300^\circ$ c) $\cos(-135)^\circ$ d) $\tan 225^\circ$

(a) $\cos 150^\circ$

$= -\cos 30^\circ$

$= -\frac{\sqrt{3}}{2}$

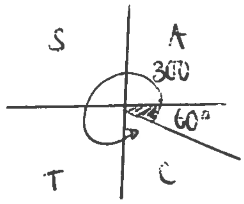


ra 30°
cos negative

(b) $\sin 300^\circ$

$= -\sin 60^\circ$

$= -\frac{\sqrt{3}}{2}$

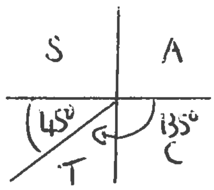


ra 60°
sin negative

(c) $\cos(-135^\circ)$

$= -\cos 45^\circ$

$= -\frac{1}{\sqrt{2}}$

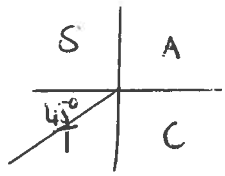


ra 45°
cos negative

(d) $\tan 225$

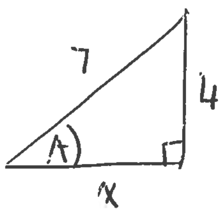
$= \tan 45$

$= 1$



tan positive
ra 45°

2) Given that $\sin A = \frac{4}{7}$ find the exact value of cosA and sinA.



$x^2 = 7^2 - 4^2$

$x^2 = 33$

$x = \sqrt{33}$

$\cos A = \frac{adj}{hyp}$

$\cos A = \frac{\sqrt{33}}{7}$

$\tan A = \frac{opp}{adj}$

$\tan A = \frac{4}{\sqrt{33}}$.

Page 28 Exercise 2A

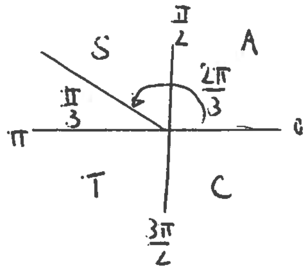
Examples

1) Find the exact values of :- a) $\cos\frac{2\pi}{3}$ b) $\sin^2 300°$ c) $\tan\frac{11\pi}{4}$

(a) $\cos\frac{2\pi}{3}$

$= -\cos\frac{\pi}{3}$

$= -\frac{1}{2}$



ra $\frac{\pi}{3}$

cos negative.

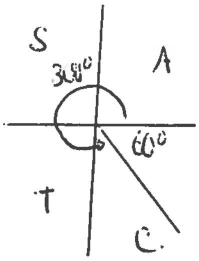
(b) $\sin^2 300°$

$= (\sin 300°)^2$

$= (-\sin 60°)^2$

$= \left(-\frac{\sqrt{3}}{2}\right)^2$

$= \frac{3}{4}$



ra 60°

Sin negative

(c) $\tan\frac{11\pi}{4}$

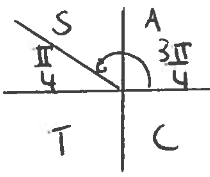
$= \tan\frac{3\pi}{4}$

$= -\tan\frac{\pi}{4}$

$= -1$

$\frac{11\pi}{4} = 2\frac{3}{4}\pi = 2\pi + \frac{3\pi}{4}$

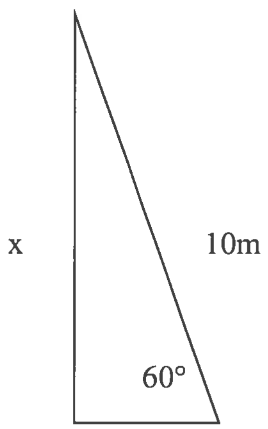
↑ full turn.



ra $\frac{\pi}{4}$

tan negative.

2) Calculate the exact length of the side marked x in the triangle.



$\sin 60° = \frac{x}{10}$

$x = 10\sin 60°$

$= 10 \times \frac{\sqrt{3}}{2}$

$x = 5\sqrt{3}$ m.

3) Find $\cos\frac{\pi}{3} - 2\sin^2\frac{\pi}{4}$

$\cos\frac{\pi}{3} - 2\sin^2\frac{\pi}{4}$

$= \frac{1}{2} - 2\left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2} - 2\times\frac{1}{2}$

$= 0.$

(p28 Ex 2A) & p31 Ex 2B Q4

complete.

AS 1.2 Applying Trigonometric Skills to Manipulating Expressions

Addition Formulae (Given but best learnt)

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Examples

1) (a) Find the exact value of sin 75° (b) Simplify sin55cos35 +cos55sin35

(a) $\sin 75^\circ = \sin(45+30)^\circ$

$= \sin 45\cos 30 + \cos 45\sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

(b) $\sin 55\cos 35 + \cos 55\sin 35$

$= \sin(55+35)$

$= \sin 90$

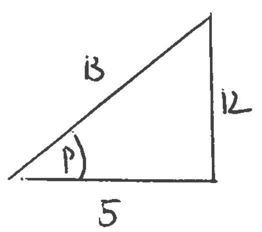
$= 1$

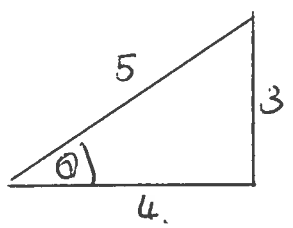
p36 Ex2D Q ① ②

2) If P and Q are acute angles such that $\sin P = \frac{12}{13}$ and $\sin Q = \frac{3}{5}$, show that

$$\sin(P+Q) = \frac{63}{65}$$

When given trig ratios draw triangles and use Pythagoras *





By Pythagoras

$x^2 = 13^2 - 12^2$

$= 25$

$x = 5$

$\sin(P+Q) = \sin P\cos Q + \cos P\sin Q$

$= \frac{12}{13} \times \frac{4}{5} + \frac{5}{13} \times \frac{3}{5}$

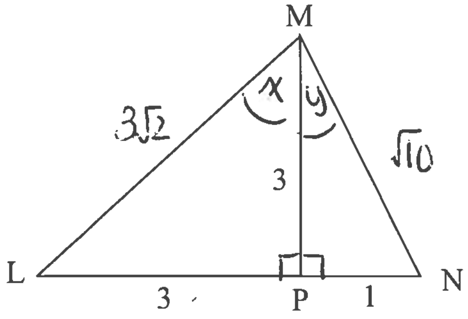
$= \frac{48}{65} + \frac{15}{65}$

$= \frac{63}{65}$ as required.

3) For the diagram below show that $\cos L\hat{M}N = \frac{\sqrt{5}}{5}$

$LM^2 = 3^2 + 3^2$
$LM = \sqrt{18}$
$= 3\sqrt{2}$

$MN^2 = 3^2 + 1^2$
$MN = \sqrt{10}$



$$\cos L\hat{M}N = \cos(L\hat{M}P + P\hat{M}N)$$
$$= \cos(x+y)$$
$$= \cos x \cos y - \sin x \sin y$$
$$= \frac{3}{3\sqrt{2}} \times \frac{3}{\sqrt{10}} - \frac{3}{3\sqrt{2}} \times \frac{1}{\sqrt{10}}$$
$$= \frac{9-3}{3\sqrt{20}}$$
$$= \frac{6}{3\sqrt{20}}$$
$$= \frac{2}{\sqrt{4 \times 5}}$$
$$= \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{\sqrt{5}}{5}$$ as required.

p33 Ex 2C & p36 Ex 2D

Double Angle Formulae (Given but best learnt)

$$\sin 2A = 2\sin A\cos A$$

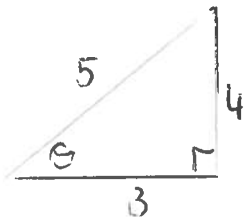
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

Examples

1) Given that θ is acute with $\tan\theta = \frac{4}{3}$, calculate the exact values of

a) $\sin 2\theta$ b) $\cos 2\theta$

draw triangle



(a) $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{4}{5} \times \frac{3}{5}$

$= \frac{24}{25}$

(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

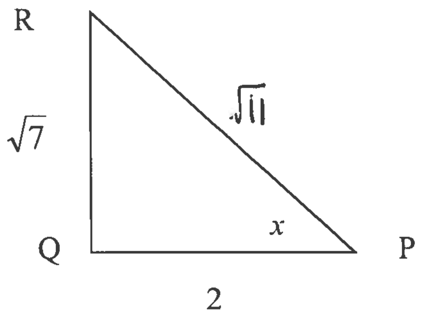
$= \frac{9}{25} - \frac{16}{25}$

$= \frac{-7}{25}$

can use any of the three formula

<u>Note</u> negative so 2θ must be obtuse.

2) Using $\triangle PQR$ find the **exact** value of cos2x



$RP^2 = 7 + 4$
$= 11$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(\frac{2}{\sqrt{11}}\right)^2 - \left(\frac{\sqrt{7}}{\sqrt{11}}\right)^2$$
$$= \frac{4}{11} - \frac{7}{11}$$
$$= -\frac{3}{11}$$

3) Find the exact value of $1 - 2\sin^2 22 \cdot 5°$

$$1 - 2\sin^2 22.5°$$
$$= \cos(2 \times 22.5°)$$
$$= \cos 45$$
$$= \frac{1}{\sqrt{2}}$$

p40 Ex 2E & p42 Ex 2F

Trigonometric Identities

$\sin^2 x + \cos^2 x = 1$ $\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$	$\tan x = \frac{\sin x}{\cos x}$
$\sin(-A) = -\sin A$ $\cos(-A) = \cos A$ $\tan(-A) = -\tan A$	$\sin(90 - A)° = \cos A°$ $\cos(90 - A)° = \sin A°$
$\sin(A + B) = \sin A\cos B + \cos A\sin B$ $\sin(A - B) = \sin A\cos B - \cos A\sin B$ $\cos(A + B) = \cos A\cos B - \sin A\sin B$ $\cos(A - B) = \cos A\cos B + \sin A\sin B$	
$\sin 2A = 2\sin A\cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $= 2\cos^2 A - 1$ $= 1 - 2\sin^2 A$	$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

Example

1. Prove that $\cos^2 x\,(1 - \tan^2 x) = 1 - 2\sin^2 x$

$$
\begin{aligned}
LHS &= \cos^2 x\,(1 - \tan^2 x) \\
&= \cos^2 x - \cos^2 x \tan^2 x \\
&= \cos^2 x - \cos^2 x \frac{\sin^2 x}{\cos^2 x} \\
&= \cos^2 x - \sin^2 x \\
&= (1 - \sin^2 x) - \sin^2 x \\
&= 1 - 2\sin^2 x \\
&= RHS \quad \text{as required}
\end{aligned}
$$

2. Prove that $\cos(90 + x)° = -\sin x°$

$$
\begin{aligned}
LHS &= \cos(90 + x) \\
&= \cos 90 \cos x - \sin 90 \sin x \\
&= 0 - \sin x \\
&= -\sin x \\
&= RHS \quad \text{as required}
\end{aligned}
$$

$\cos 90 = 0$
$\sin 90 = 1$ from graphs

11 12

3. Prove that $\dfrac{\sin(a+b)}{\cos a \cos b} = \tan a + \tan b$

$$\begin{aligned} LHS &= \frac{\sin(a+b)}{\cos a \cos b} \\ &= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b} \\ &= \frac{\sin a \cos b}{\cos a \cos b} + \frac{\cos a \sin b}{\cos a \cos b} \\ &= \tan a + \tan b \\ &= RHS \quad \text{as required.} \end{aligned}$$

4. Express $4\cos^2 x$ in terms of $\cos 2x$

$$\begin{aligned} &4\cos^2 x \\ &= 4\left(\tfrac{1}{2}(1+\cos 2x)\right) \\ &= 2(1+\cos 2x) \end{aligned}$$

5. Prove that $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$

$$\begin{aligned}
LHS &= \sin^4 x \\
&= (\sin^2 x)^2 \\
&= \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \\
&= \tfrac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \\
&= \tfrac{1}{4}\left(1 - 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\right) \\
&= \tfrac{1}{4}\left(1 - 2\cos 2x + \tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) \\
&= \tfrac{1}{4}\left(\tfrac{1}{2}\cos 4x - 2\cos 2x + \tfrac{3}{2}\right) \\
&= \tfrac{1}{8}(\cos 4x - 4\cos 2x + 3) \\
&= RHS \quad \text{as required.}
\end{aligned}$$

p45 Ex 2G

~~13~~ 14

The Wave Function - Combining Sine and Cosine Functions

Sine and cosine functions may be combined and we easily see that the result is a wave in the form of a sine or cosine wave: e.g. $k\sin(x \pm \alpha)$ or $k\cos(x \pm \alpha)$.

In summary :-

$$a\cos x + b\sin x = k\cos(x \pm \alpha)$$

or $$a\cos x + b\sin x = k\sin(x \pm \alpha) \qquad k > 0$$

Example

1) Write $3\cos x° - 4\sin x°$ in the form $k\cos(x-\alpha)°$ for $0 \le \alpha \le 360$

* Learn
* method.

$3\cos x - 4\sin x = k\cos(x-\alpha)$
$= k(\cos x\cos\alpha + \sin x\sin\alpha)$
$3\cos x - 4\sin x = k\cos\alpha\cos x + k\sin\alpha\sin x$

Comparing sides.

$k\cos\alpha = 3$ (*) — infront of cosx
$k\sin\alpha = -4$ — infront of sinx

Square and Add

$k^2\cos^2\alpha + k^2\sin^2\alpha = 3^2 + (-4)^2$
$k^2(\cos^2\alpha + \sin^2\alpha) = 9 + 16$
$k^2 = 25$
$k = 5$

$(k > 0)$
* always take $+\sqrt{}$

Divide

Sine divided by cos to give tan

$\frac{k\sin\alpha}{k\cos\alpha} = \frac{-4}{3}$

$\tan\alpha = \frac{-4}{3}$

$\alpha = 306.9°$

ra $\tan^{-1}\frac{4}{3}$
$= 53.1°$

S	A
T	C ✓

cos + from
sin − (*)
tan −

So $3\cos x - 4\sin x = 5\cos(x - 306.9°)$

2) (a) Write $\cos x - \sqrt{3}\sin x$ in the form $k\sin(x - \alpha)$, *for* $0 \le \alpha \le 2\pi$ radians

(b) Hence find the maximum value of

$$4\cos x - 4\sqrt{3}\sin x + 1$$

and value of x for which this occurs. $(0 \le x \le 2\pi)$

(a)
$$\cos x - \sqrt{3}\sin x = k\sin(x-\alpha)$$
$$= k\sin x\cos\alpha - k\cos x\sin\alpha$$
$$= (k\cos\alpha)\sin x - (k\sin\alpha)\cos x$$
$$= -\sqrt{3}\,\sin x + \cos x$$

Comparing gives.

$$k\cos\alpha = -\sqrt{3}$$
$$-k\sin\alpha = 1 \Rightarrow k\sin\alpha = -1$$

Square and add
$$k^2 = (-\sqrt{3})^2 + (-1)^2$$
$$k^2 = 4$$
$$k = 2$$

Divide
$$\tan\alpha = -\frac{1}{-\sqrt{3}}$$
$$\tan\alpha = \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{7\pi}{6}$$

cos −
sin −
tan +

ra $\frac{\pi}{6}$.

S	A
T	C

$$\pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

So $\cos x - \sqrt{3}\sin x = 2\sin\left(x - \frac{7\pi}{6}\right)$

(b)
$$4\cos x - 4\sqrt{3}\sin x + 1 = 4(\cos x - \sqrt{3}\sin x) + 1$$
$$= 8\sin\left(x - \frac{7\pi}{6}\right) + 1$$

max value $8+1$
$= 9$

when angle $= \frac{\pi}{2}$
$$x - \frac{7\pi}{6} = \frac{\pi}{2}$$
$$x = \frac{10\pi}{6}$$
$$x = \frac{5\pi}{3}$$

p52 Ex 2I Q1 - 6

Other Forms

You should be able to write combined functions, $a\cos x + b\sin x$, in **ALL** of the four formats shown above, and be comfortable in both degrees or radians.

You should also be able to work with multiple angles - e.g. $a\cos 2x + b\sin 2x$

Example

Write $\cos 2x + \sqrt{3}\sin 2x$ in the form $k\sin(2x+\alpha)$ for $0 \le \alpha \le 2\pi$

radians.

note '2x matches.'

$$\cos 2x + \sqrt{3}\sin 2x = k\sin(2x+\alpha)$$
$$= k\sin 2x\cos\alpha + k\cos 2x\sin\alpha$$
$$= \underline{k\cos\alpha}\sin 2x + \underline{k\sin\alpha}\cos 2x$$

Compare $= \underline{\sqrt{3}}\sin 2x + \_\cos 2x$

So $k\cos\alpha = \sqrt{3}$

$k\sin\alpha = 1$

Square and add $k^2 = 3+1$

$k = 2$

Divide $\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$$\cos 2x + \sqrt{3}\sin 2x = 2\sin\left(2x + \frac{\pi}{6}\right)$$

p52 Ex 2I Q7 - 9