AS 1.1 Applying Algebraic Skills to Logarithms and Exponentials

Exponential Functions and their Graphs

$y = a^x$ where $0 < a < 1$ or $a > 1$,

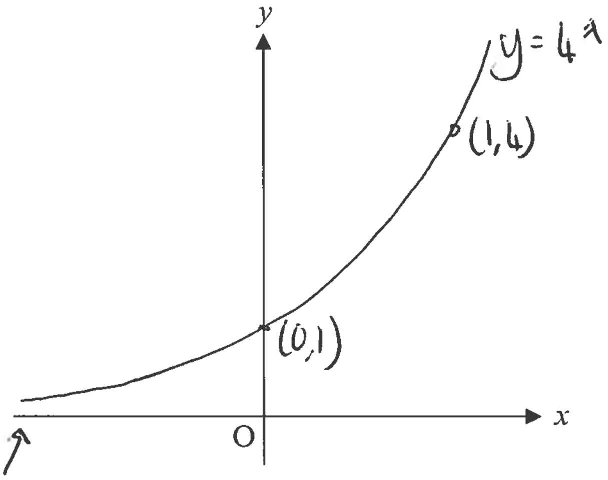
note
when $a = 1$, $y = 1$
which is a straight line parallel to x-axis

is an exponential function with base a and exponent (or index) x.

If $a > 1$ the graph of the exponential function increases rapidly. This is called a **growth function**.

For the graph of $y = 4^x$

- y is always positive
- it never touches the x-axis
- it crosses y-axis at (0 , 1)
- it goes through the point (1 , 4)
- it is increasing

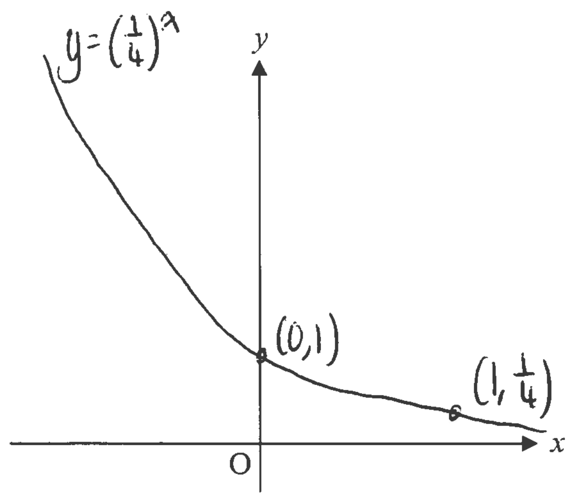


The x-axis is an asymptote
The graph gets closer and closer to it but never touches or crosses it

If $0 < a < 1$ the graph of the exponential function decreases rapidly. This is called a **decay function**.

For the graph of $y = \left(\frac{1}{4}\right)^x$

- y is always positive
- it never touches the x-axis
- it crosses y-axis at (0 , 1)
- it goes though the point $(1, \frac{1}{4})$
- it is decreasing



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A Special Growth Function

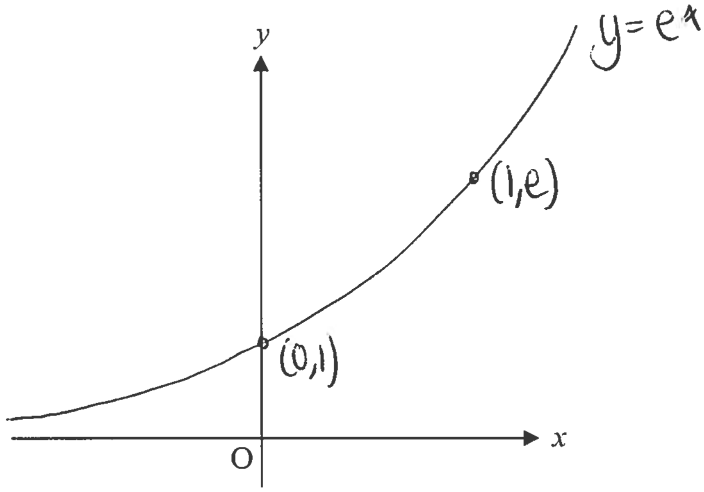
The function

$$y = e^x$$

is called the exponential function to the base e and is a special exponential function which plays an important role in mathematics.

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$
$$= 2.71828......$$

Since $e > 1$, $y = e^x$ is a growth function.



Note $y = e^x$ is sometimes written $y = \exp(x)$

Examples

1. Calculate y to 3 significant figures.

(a) $y = 5e^2$

y= 36.9 (3s.f)

(b) $y = 2e^{-0.2 \times 6}$

y= 0.602 (3s.f)

Use the $e^{\square}$ button on your calculator

2. The number of bacteria of a particular strain is given by the formula

$$B(t) = B_o e^{0.09t}$$

where B_0 is the initial number of bacteria and t is the number of hours since the experiment began.

(a) If $B_0 = 2000$, calculate the number of bacteria present after 5 hours.

(b) How many whole ~~years~~ hours. would it take for the number of bacteria to more than double?

(a) $B(t) = B_0 e^{0.09t}$ $\quad B_0 = 2000 \quad t = 5$

$B(5) = 2000\, e^{0.09 \times 5}$

$= 3137$ (to nearest whole number)

(b) For bacteria to double need $B(t) > 4000$

* Use trial and error

We will find a better method for answering this question later.

$B(6) = 2000 \times e^{0.09 \times 6}$

$= 3432.$

$B(7) = 3755$

$B(8) = 4109$

It will take 8 hours

Page 4 Exercise 1A

Logarithmic Functions

The logarithmic function is the inverse of the exponential function.

$$\boxed{\text{If} \quad y = a^x \quad \text{then} \quad x = \log_a y}$$

base a $\qquad$ base a

** LEARN

Note y is always positive so you cannot take the log of a negative number **

Examples

1. Express each exponential function as a logarithm.

(a) $4^2 = 16$ $\qquad$ (b) $y = 5^4$ $\qquad$ (c) $p = s^{3t}$

base 4 $\qquad$ base 5 $\qquad$ base s

$2 = \log_4 16$ $\qquad$ $4 = \log_5 y$ $\qquad$ $3t = \log_s p.$

2. Express each log function as an exponential.

(a) $3 = \log_5 125$

base 5

$5^3 = 125$

(b) $y = \log_{10} 200$

base 10

$10^y = 200$

(c) $3t = \log_{10} p$

base 10

$10^{3t} = p$

3. Calculate the value of a in each question.

(a) $a = \log_3 9$

base

$3^a = 9$

$a = 2$

(b) $\log_a 16 = 4$

$16 = a^4$

$a = 2$

Change logs to indices to work out a missing value.

(c) $\log_2 a = 3$

$a = 2^3$

$a = 8$

(d) $a = \log_{64} 16$

$16 = 64^a$

write in terms of the same base

$2^4 = (2^6)^a$

$2^4 = 2^{6a}$

so $4 = 6a$

$a = \frac{2}{3}$

(e) $a = \log_{\frac{1}{3}} 81$

$81 = \left(\frac{1}{3}\right)^a$

$3^4 = (3^{-1})^a$

$3^4 = 3^{-a}$

so $-a = 4$

$a = -4$.

Page 6 Exercise 1B

The Laws of Logarithms

It can be shown that for logs of the **same base**, the following relationships occur. These are known as the **laws of logs.**

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a x + \log_a y = \log_a (xy)$
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
- $\log_a x^n = n \log_a x$

* LEARN

Examples

1. Simplify the following as far as possible

(a) $\log_2 4 + \log_2 8$

$= \log_2 32$
$= \log_2 2^5$
$= 5\log_2 2 = 5$

write as power of base if possible.

since $\log_2 2 = 1$

(b) $\log_6 9 + \log_6 8 - \log_6 2$

$= \log_6 \frac{9 \times 8}{2}$

$= \log_6 36 = \log_6 6^2 = 2\log_6 6$
$= 2$

(c) $\log x^5 - \log x + \log \frac{1}{x^2}$

$= \log \frac{x^5 \times \frac{1}{x^2}}{x}$

$= \log x^2$

$= 2\log x$

note if we write log with no base indicated this usually means $\log_{10}$

2. Evaluate

(a) $\frac{1}{3}\log_4 64$

$= \log_4 64^{\frac{1}{3}}$
$= \log_4 4$
$= 1$

or $\frac{1}{3}\log_4 4^3$
$= 3 \times \frac{1}{3}\log_4 4$
$= 1.$

(b) $\log_y y^6$

$= 6\log_y y$
$= 6$

since $\log_y y = 1$

3. Simplify $2\log_{10}5 + 5\log_{10}2 + \log_{10}3^3 - \log_{10}6^3$

$= \log_{10}5^2 + \log_{10}2^5 + \log_{10}3^3 - \log_{10}6^3$

numbers at front go inside log as a power first

$= \log_{10}\dfrac{5^2 \times 2^5 \times 3^3}{6^3}$

combine adding logs -> multiply
subtracting log -> divide

$= \log_{10}100$

$= \log_{10}10^2$

$= 2\log_{10}10 = 2.$

4. Simplify $\log x^3 + \log\frac{1}{x} - 2\log x$

$= \log x^3 + \log\frac{1}{x} - \log x^2$

$= \log\dfrac{x^3 \times \frac{1}{x}}{x^2}$

$= \log 1$

$= 0$

5. If $\log_a y = \log_a 8 + 3\log_a x$ express y in terms of x.

$\log_a y = \log_a 8 + \log_a x^3$

$\log_a y = \log_a 8x^3$

$y = 8x^3$

Page 9 Exercise 1C

The Natural Logarithm

The inverse of the exponential function $y = e^x$ is called the natural logarithm

$$y = \log_e x$$

$y = \log_e x$ can also be written as $\ln x$

Note $\log_{10} x$ is written as $\log x$ on the calculator

$\log_e x$ is written as $\ln x$

Examples

1. Express in exponential form

(a) $y = \log_e 5$

base.

$5 = e^y$

(b) $y = \log_e 3t$

$3t = e^y$

2. Express in log form

(a) $y = e^7$

$7 = \log_e y$

$7 = \ln y$

(b) $k = e^{3r}$

$3r = \log_e k$

$3r = \ln k$.

3. Simplify $2\ln e^2 + 3\ln e - \ln e^6$

$= \ln e^4 + \ln e^3 - \ln e^6$

$= \ln \frac{e^4 \times e^3}{e^6}$

$= \ln e$

$= 1$

since $\log_e e = 1$

Page 11 Exercise 1D

Using Logs in Solving Equations

Type 1 Types which can be written log (something) = number

Method Change to exponentials

Example

1. Solve for $x > 0$ $\ln x = 5 \cdot 6$

base e

$x = e^{5.6}$

$x = 270.4$ (1 d.p)

use calculator

2. Solve $\log_2 x + \log_2 (x-2) = 3$

combine to give one log

change to indices

$\log_2 x(x-2) = 3$

$x(x-2) = 2^3$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\underline{\underline{x = 4}}$ or $\cancel{x = -2}$ NA

Remember to check that values of x are suitable → can't have the log of a negative.

3. Solve $\log_3 2x + \log_5 25 = 4$

bases different so can't combine but spot $25 = 5^2$

$\log_3 2x + \log_5 5^2 = 4$

$\log_3 2x + 2\log_5 5 = 4$

$\log_3 2x + 2 = 4$

$\log_3 2x = 2$

$2x = 3^2$

$\underline{\underline{x = \frac{9}{2}}}$.

Page 12 Exercise 1E Questions 1 ,2 and 5
Page 15 Exercise 1G

<u>Type 2</u> Equations that can be written log(something) = log(something)

Method If $\log x = \log y$ then $x = y$

<u>Example</u>

1. Solve $\log_a x + \log_a 5 = \log_a 35$

combine logs on RHS

$\log_a 5x = \log_a 35$

so $5x = 35$

$\underline{\underline{x = 7}}$

2. Solve $\log_a (2x+1) + \log_a (3x-10) = \log_a 11x$

$\log_a (2x+1)(3x-10) = \log_a 11x$

$(2x+1)(3x-10) = 11x$

$6x^2 - 17x - 10 = 11x$

$6x^2 - 28x - 10 = 0$

$3x^2 - 14x - 5 = 0$

$(3x+1)(x-5) = 0$

$x = -\frac{1}{3}$ NA. $x = 5$

Solution $\underline{x = 5}$

(since $x = -\frac{1}{3}$ gives log (negative no.))

AS 1.1 Applying Algebraic Skills to Logarithms and Exponentials

Note The domain means the x numbers used.
If the question asks about a suitable domain it is asking you to check that your x numbers do not lead to finding the log of a negative number which is not possible.

Page 13 Exercise 1F

Type 3 Unknown is a power (exponent)

Method Either change straight to logs or take logs of both sides

Example

1 Solve (a) $5^x = 50$ (b) $e^{2x+1} = 50$

(a)

Method ①

$5^x = 50$

$x = \log_5 50$

$= 2.43$ (3s.f)

Method ②

$\log 5^x = \log 50$

$x \log 5 = \log 50$

$x = \frac{\log 50}{\log 5}$

$= 2.43$ (3s.f)

(b)

Method ①

$2x+1 = \ln 50$

$x = \frac{\ln 50 - 1}{2}$

$x = 1.46$ (3s.f)

Method ②

$\ln e^{2x+1} = \ln 50$

$(2x+1)\ln e = \ln 50$

$2x+1 = \ln 50$

$x = \frac{\ln 50 - 1}{2}$

$= 1.46$ (3s.f)

(c) $10e^{1.5t} = 200$

* Divide by 10 first

$e^{1.5t} = 20$.

Method ①

$1.5t = \ln 20$

$t = \frac{\ln 20}{1.5}$

$t = 2.00$ (3s.f)

Method ②

$\ln e^{1.5t} = \ln 20$

$1.5t \ln e = \ln 20$

$1.5t = \ln 20$

$t = \frac{\ln 20}{1.5}$

$= 2.00$ (3s.f)

Page 12 Exercise 1E Questions 3 and 4

Applications of Exponential Functions

Examples

1. A young tree, supplied by a nursery, is 4 metres tall but has annual growth of 5% of its height at the start of each year.
   How long does it take the tree to double in height ?

5% increase $\Rightarrow$ 105% altogether
$= 1.05$

Let n = number of years.
new height $= 4 \times 1.05^n$

We want $4 \times 1.05^n = 8$
$1.05^n = 2$
$n = \log_{1.05} 2$
$= \underline{\underline{14.2 \text{ years}}}$ (3 s.f)

2. The air pressure of a life raft falls according to the formula $P_t = P_0 e^{-kt}$, where P_t is the pressure at time t hours and k is a constant.

(a) At time zero the pressure is 80 units. 12 hours later it is 60 units.
Find the value of k to two significant figures.

(b) When the pressure is below 40 units the raft is unsafe. From time zero, for how long is the raft safe to use?

(a) When $t=0$ $P_0 = 80$
When $t=12$ $P_{12} = 60$

In $P_t = P_0 e^{-kt}$
$60 = 80e^{-12k}$
$0.75 = e^{-12k}$
$-12k = \ln 0.75$
$k = \frac{\ln 0.75}{-12}$
$k = 0.024$ (2 s.f)

(b) $P_t = 40$ $k = 0.024$
so $40 = 80e^{-0.024t}$
$0.5 = e^{-0.024t}$
$-0.024t = \ln 0.5$
$t = \frac{\ln 0.5}{-0.024}$
$= 29$ hours.
(2 s.f)

The raft is safe to use 29 hours.

3. U^{235} is a radioactive isotope of uranium. It decays into lead according to the law $m = m_0 e^{kt}$ where m_0 is the mass of U^{235} originally present and m is the mass present after t years.

   a) The half-life of U^{235}, i.e. the time taken for half of the isotope to decay, is $7\text{x}10^8$ years. find the value of k correct to two significant figures.
   b) A sample of rock contains 20mg of U^{235}. How long will it be before this is reduced by 0.5mg?
   c) The age of the earth is estimated to be $5.25\text{x}10^9$ years. What fraction of the U^{235} present at its formation is still around today?

(a) $m = \frac{1}{2} m_0$ when $t = 7\times10^8$

We have
$$m = m_0 e^{kt}$$
$$\tfrac{1}{2} m_0 = m_0 e^{(7\times10^8)k}$$
$$0.5 = e^{(7\times10^8)k}.$$
$$(7\times10^8)k = \ln 0.5$$
$$k = \frac{\ln 0.5}{7\times10^8}$$
$$k = -9.9\times10^{-10} \quad (2 \text{ s.f})$$

(b) $m_0 = 20$, $m = 19.5$ $k = -9.9\times10^{-10}$

So $m = m_0 e^{kt}$ gives
$$19.5 = 20e^{(-9.9\times10^{-10})t}$$
$$\frac{19.5}{20} = e^{(-9.9\times10^{-10})t}$$
$$(-9.9\times10^{-10})t = \ln\left(\frac{19.5}{20}\right)$$
$$t = \frac{\ln 0.975}{-9.9\times10^{-10}}$$
$$t = 25573543 \text{ years}$$
$$= 2.6\times10^7 \text{ years} \quad (2 \text{ s.f})$$

(c) $t = 5.25\times10^9$ so in
$$m = m_0 e^{kt}$$
$$\frac{m}{m_0} = e^{-9.9\times10^{-10}\times5.25\times10^9}$$
$$\frac{m}{m_0} = 0.0055$$
$$= \frac{55}{10000}$$
$$= \frac{11}{2000}$$

Page 16 Exercise 1H

Applications of Logarithms

Example
Two sound intensities P_1 and P_2 are said to differ by n decibels when

$$n = 10\log_{10}\frac{P_2}{P_1}$$

where P_1 and P_2 are measured in phons and $P_2 > P_1$.
Rustling leaves have typical sound intensity of 30 phons. If the sound intensity of a fire alarm is 6.5 decibels greater that rustling leaves, what is the sound intensity of the fire alarm siren?

$P_1 = 30$
$P_2 = ?$
$n = 6.5$

We have $n = 10\log_{10}\frac{P_2}{P_1}$

$6.5 = 10\log_{10}\frac{P_2}{30}$

$0.65 = \log_{10}\frac{P_2}{30}$

Change to Indices

$\frac{P_2}{30} = 10^{0.65}$

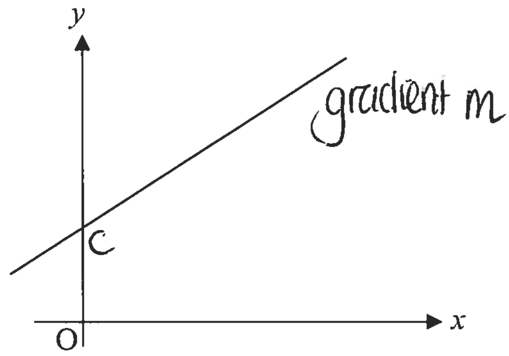
$P_2 = 30 \times 10^{0.65}$

$= \underline{134 \text{ phons}}$ (3 s.f)

Page 18 Exercise 1I

Interpreting Experimental Data

Data obtained by experiment is often used to determine a relationship or formulae between the variables involved.



This graph shows a linear (straight line) relationship between y and x.

The formula is

$y = mx + c$

where m is the gradient of the line
and c is the y intercept.

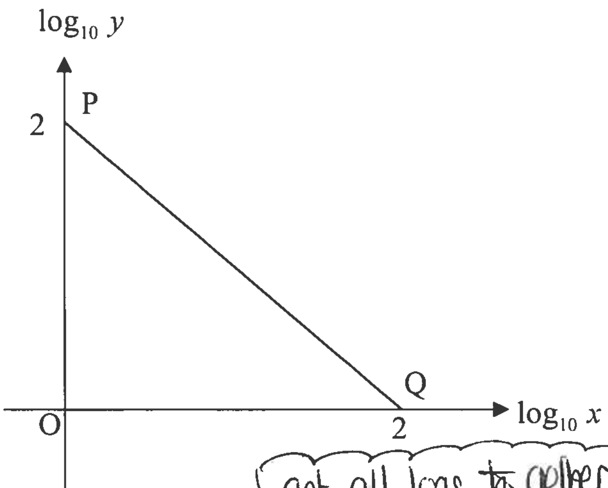
If instead of a linear relationship between the variables there is an exponential growth or decay function then logarithms can be used to find the equation.

If logy plotted against logx gives a straight line then y and x are related by the formula $y = ax^b$ where a and b are constants that can be found from the graph.

If logy plotted against x gives a straight line then y and x are related by the formula $y = ab^x$ where a and b are constants that can be found from the graph.

Example

1. Collected data is processed and plotted to obtain the graph shown below. Show that the graph represents a function of the form $y = ax^b$.
   Determine the values of a and b.

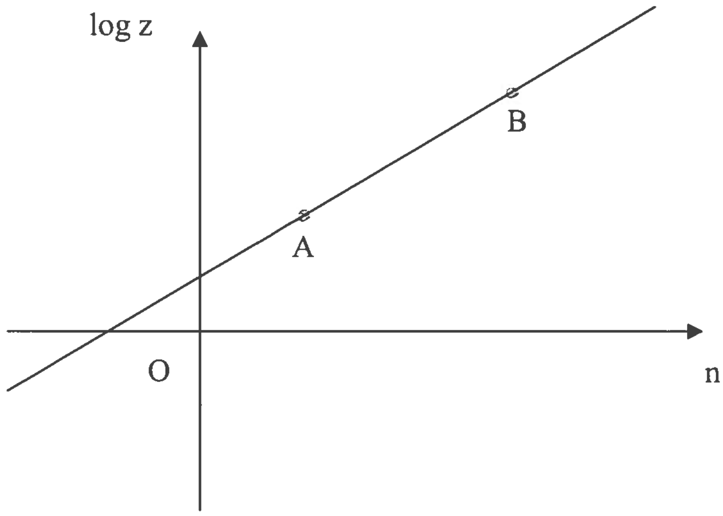


Straight line
$y = mx + c$.
but here.
$\log_{10} y = m \log_{10} x + c$.
points $(2,0)$ $(0,2)$
$m = \frac{2-0}{0-2}$ $c = 2$
$= -1$
So $\log_{10} y = -\log_{10} x + 2$

get all logs together and combine → $\log_{10} y + \log_{10} x = 2$
$\log_{10} yx = 2$
change to indices → $yx = 10^2$
$yx = 100$
$y = \frac{100}{x} = 100x^{-1}$

Old Higher book: D2/2 EX15T 2

2. Data from an experiment gives the following graph.



The points A(2.5, 1.456) and B(8.0, 3.09) are 2 points on the straight line. Determine the specific relationship between z and n.

Straight line $y = mx + c$.

Here $\log z = mn + c$.

and $m = \dfrac{3.09 - 1.456}{8.0 - 2.5}$

$= 0.30$

$\log z = 0.30n + c$.

To find c substitute point A (or B) $(2.5, 1.456)$ — n, $\log z$.

$1.456 = 0.30 \times 2.5 + c$

$c = 0.71$

so $\log z = 0.30n + 0.71$

Change to indices

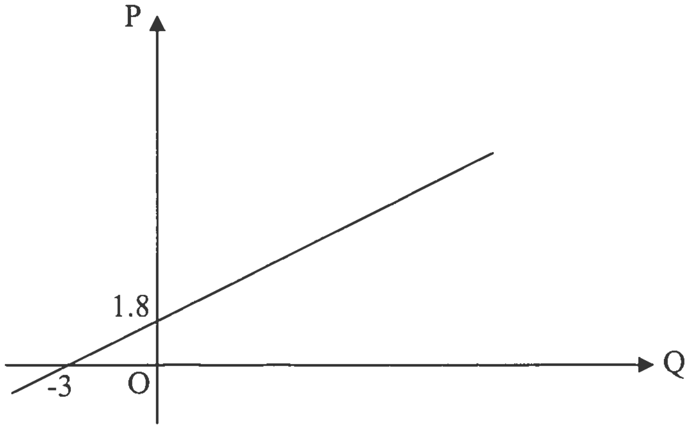
$z = 10^{0.30n + 0.71}$

$z = 10^{0.30n} \times 10^{0.71}$

$z = (10^{0.3})^n \times 5.13$

$\underline{\underline{z = 5.13 \times 2^n}}$ (since $10^{0.3} = 2.00$ (3sf))

3. The results from an experiment give rise to the graph shown.



(a) Write down the equation of the line in terms of P and Q.

(b) Given that P = ln$p$ and Q = ln$q$, show that p and q satisfy the relationship $p = aq^b$, and determine the values of a and b.

(a) Straight line $y = mx + c$.

Here $P = mQ + c$.

$m = \dfrac{1.8 - 0}{0 - (-3)}$ $c = 1.8$

$= 0.6$

so $P = 0.6Q + 1.8$

(b) Put $P = \ln p$ $Q = \ln q$

$\ln p = 0.6 \ln q + 1.8$

Get logs together

$\ln p - 0.6 \ln q = 1.8$

$\ln p - \ln q^{0.6} = 1.8$

$\ln \dfrac{p}{q^{0.6}} = 1.8$

Change to indices $\dfrac{p}{q^{0.6}} = e^{1.8}$

$p = e^{1.8} q^{0.6}$

$p = 6.05 q^{0.6}$ (3 s.f)

So $\underline{\underline{a = 6.05}}$ $\underline{\underline{b = 0.6}}$

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