X100/12/13

NATIONAL QUALIFICATIONS 2015

WEDNESDAY, 20 MAY
10.50 AM – 12.00 NOON

MATHEMATICS
HIGHER
Paper 2

Read carefully

Calculators may be used in this paper.

Full credit will be given only to solutions which contain appropriate working.

Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. Additional space for answers is provided at the end of the answer booklet. If you use this space you must clearly identify the question number you are attempting.

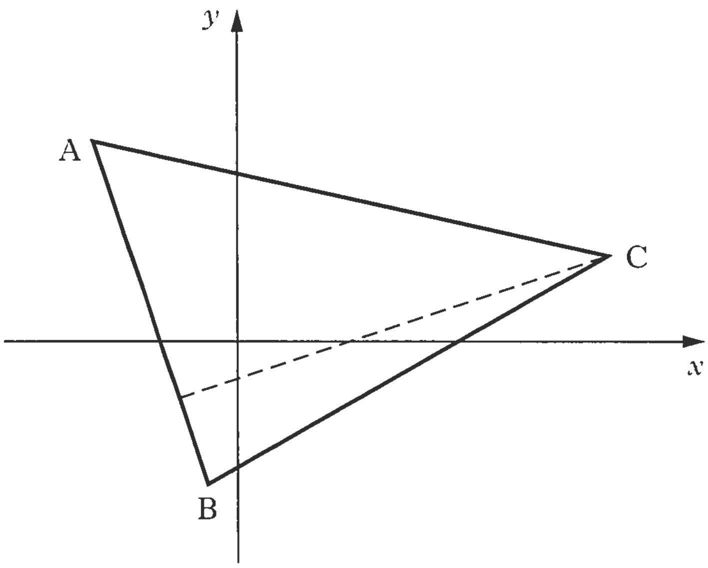
Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

ALL questions should be attempted. *Marks*

1. The vertices of triangle ABC are A(–5, 7), B(–1, –5) and C(13, 3) as shown in the diagram.

   The broken line represents the altitude from C.



   (*a*) Show that the equation of the altitude from C is $x - 3y = 4$. **4**

   (*b*) Find the equation of the median from B. **3**

   (*c*) Find the coordinates of the point of intersection of the altitude from C and the median from B. **2**

2. Functions f and g are defined on suitable domains by

$$f(x) = 10 + x \text{ and } g(x) = (1 + x)(3 - x) + 2.$$

   (*a*) Find an expression for $f(g(x))$. **2**

   (*b*) Express $f(g(x))$ in the form $p(x + q)^2 + r$. **3**

   (*c*) Another function h is given by $h(x) = \dfrac{1}{f(g(x))}$.

   What values of x cannot be in the domain of h? **2**

[Turn over

Marks

3. A version of the following problem first appeared in print in the 16th Century.

A frog and a toad fall to the bottom of a well that is 50 feet deep.

Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.

The toad climbs 13 feet each day before resting.

Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3}f_n + 32, \quad f_1 = 32$
- $t_{n+1} = \frac{3}{4}t_n + 13, \quad t_1 = 13$

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

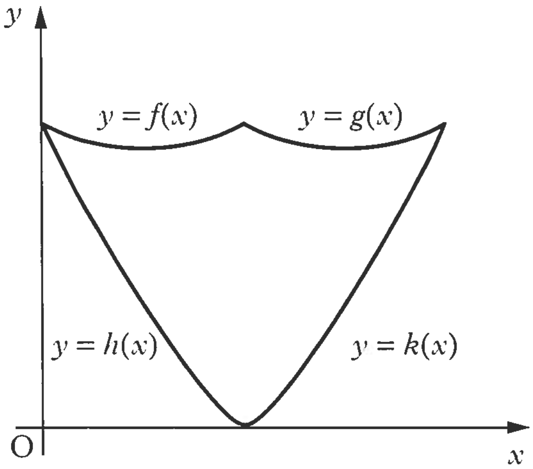
(*a*) Calculate t_2, the height of the toad at the end of the second day. **1**

(*b*) Determine whether or not either of them will eventually escape from the well. **5**

Marks

4. A wall plaque is to be made to commemorate the 150th aniversary of the publication of "*Alice's Adventures in Wonderland*".

The edges of the wall plaque can be modelled by parts of the graphs of four quadratic functions as shown in the sketch.



- $f(x) = \frac{1}{4}x^2 - \frac{1}{2}x + 3$
- $g(x) = \frac{1}{4}x^2 - \frac{3}{2}x + 5$
- $h(x) = \frac{3}{8}x^2 - \frac{9}{4}x + 3$
- $k(x) = \frac{3}{8}x^2 - \frac{3}{4}x$

(*a*) Find the x-coordinate of the point of intersection of the graphs with equations $y = f(x)$ and $y = g(x)$. **2**

The graphs of the functions $f(x)$ and $h(x)$ intersect on the y-axis.

The plaque has a vertical line of symmetry. **7**

(*b*) Calculate the area of the wall plaque.

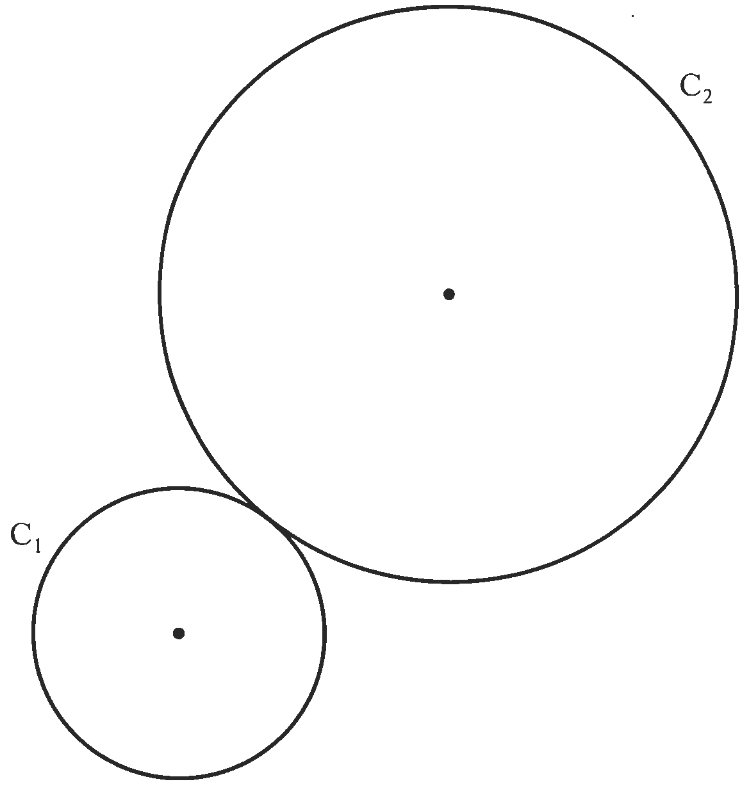
[Turn over

Marks

5. Circle C_1 has equation $x^2 + y^2 + 6x + 10y + 9 = 0$.

The centre of circle C_2 is $(9, 11)$.

Circles C_1 and C_2 touch externally.



(*a*) Determine the radius of C_2. **4**

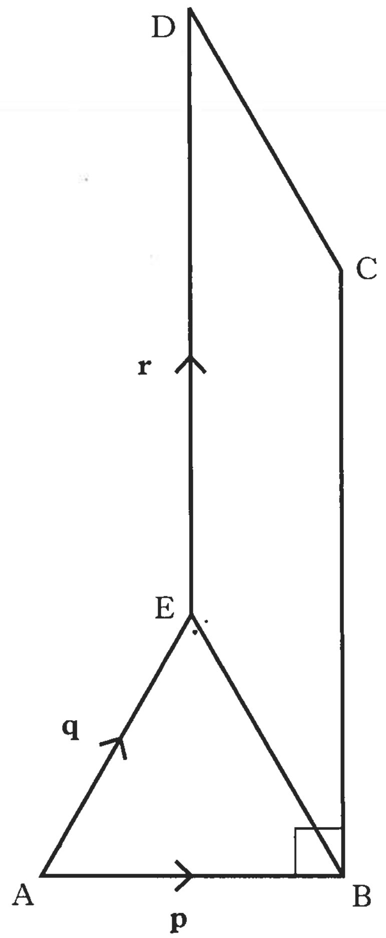
A third circle, C_3, is drawn such that:

- both C_1 and C_2 touch C_3 internally
- the centres of C_1, C_2 and C_3 are collinear.

(*b*) Determine the equation of C_3. **4**

6. Vectors **p**, **q** and **r** are represented on the diagram as shown.

- BCDE is a parallelogram
- ABE is an equilateral triangle
- $|\mathbf{p}| = 3$
- Angle ABC = 90°



(*a*) Evaluate $\mathbf{p}.(\mathbf{q} + \mathbf{r})$. **3**

(*b*) Express $\overrightarrow{EC}$ in terms of **p**, **q** and **r**. **1**

(*c*) Given that $\overrightarrow{AE}.\overrightarrow{EC} = 9\sqrt{3} - \frac{9}{2}$, find $|\mathbf{r}|$. **3**

[Turn over for Questions seven and eight on *Page eight*

Marks

7. (*a*) Find $\int (3\cos 2x + 1)\,dx$. **2**

(*b*) Show that $3\cos 2x + 1 = 4\cos^2 x - 2\sin^2 x$. **2**

(*c*) Hence, or otherwise, find $\int \left(\sin^2 x - 2\cos^2 x\right)dx$. **2**

8. The blades of a wind turbine are turning at a steady rate.

The height, h metres, of the tip of one of the blades above the ground at time, t seconds, is given by the formula

$$h = 36\sin(1{\cdot}5t) - 15\cos(1{\cdot}5t) + 65.$$

Express $36\sin(1{\cdot}5t) - 15\cos(1{\cdot}5t)$ in the form

$$k\sin(1{\cdot}5t - a), \text{ where } k > 0 \text{ and } 0 < a < \frac{\pi}{2},$$

and hence find the **two** values of t for which the tip of this blade is at a height of 100 metres above the ground during the first turn. **8**

[*END OF QUESTION PAPER*]

Marks

7. (*a*) Find $\int (3\cos 2x + 1)\, dx$. **2**

(*b*) Show that $3\cos 2x + 1 = 4\cos^2 x - 2\sin^2 x$. **2**

(*c*) Hence, or otherwise, find $\int \left(\sin^2 x - 2\cos^2 x\right) dx$. **2**

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[*END OF QUESTION PAPER*]

Higher Maths 2015 Paper 2

① A(−5, 7) B(−1, −5)

$m_{AB} = \frac{-5-7}{-1-(-5)}$

$= \frac{-12}{4}$

$= -3$ $m_{perp} = \frac{1}{3}$

Equation $y - b = m(x - a)$

$y - 3 = \frac{1}{3}(x - 13)$

$3y - 9 = x - 13$

$3y = x - 4$

$x - 3y = 4$ as required.

(b) midpt AC = M(4, 5).

$m_{MB} = \frac{5-(-5)}{4-(-1)}$

$= \frac{10}{5}$

$= 2$

Equation $y - b = m(x - a)$

$y - 5 = 2(x - 4)$

$y - 5 = 2x - 8$

$y = 2x - 3$

(c) Solve $x - 3y = 4$ ... ①

$y = 2x - 3$ ... ②

Put ② in ① $x - 3(2x - 3) = 4$

$x - 6x + 9 = 4$

$-5x = -5$

$x = 1 \Rightarrow y = -1 \qquad (1,-1).$

② (a) $f(x) = 10 + x \qquad g(x) = (1+x)(3-x)+2$

$$f(g(x)) = f((1+x)(3-x)+2)$$
$$= 10 + (1+x)(3-x)+2$$
$$= 12 + 3 + 3x - x - x^2$$
$$= 15 + 2x - x^2$$

(b) $15 + 2x - x^2.$

$$= -(x^2 - 2x) + 15$$
$$= -((x-1)^2 - (-1)^2) + 15$$
$$= -((x-1)^2 - 1) + 15$$
$$= -(x-1)^2 + 1 + 15$$
$$= -(x-1)^2 + 16$$

(c) Can't have $f(g(x)) = 0$

ie $(15 + 2x - x^2) \neq 0$

$(5 - x)(3 + x) \neq 0$

$x \neq 5$ and $x \neq -3$.

③ (a) $t_{n+1} = \frac{3}{4} t_n + 13$

$t_1 = 13$

$t_2 = \frac{3}{4} t_1 + 13$

$= \frac{3}{4} \times 13 + 13$

$= \frac{91}{4}$ feet.

(b) Find limit

$f_{n+1} = \frac{1}{3} f_n + 32$

Limit exists since $-1 < \frac{1}{3} < 1$

$L = \frac{b}{1-a}$

$= \frac{32}{1-\frac{1}{3}}$

$= 32 \div \frac{2}{3}$

$= 32 \times \frac{3}{2}$

$= 48$ feet.

Frog will not escape since it can only reach 48 feet which is less than 50 fe

$t_{n+1} = \frac{3}{4} t_n + 13$

Limit exists since $-1 < \frac{3}{4} < 1$

$L = \frac{b}{1-a}$

$= \frac{13}{1-\frac{3}{4}}$

$= \frac{13}{\frac{1}{4}}$

$= 52$ feet

The toad will escape since 52 feet > 50 feet.

(4) (a) $y = \frac{1}{4}x^2 - \frac{1}{2}x + 3$

$y = \frac{1}{4}x^2 - \frac{3}{2}x + 5$

Solve $\frac{1}{4}x^2 - \frac{1}{2}x + 3 = \frac{1}{4}x^2 - \frac{3}{2}x + 5$

$-\frac{1}{2}x + 3 = -\frac{3}{2}x + 5$

$x = 2$ $\quad y = 1 - 1 + 3$
$= 3$

$(2, 3)$

(b) area of LHS $= \int_0^2 (f(x) - h(x))\,dx$

$= \int_0^2 \left(\left(\frac{1}{4}x^2 - \frac{1}{2}x + 3\right) - \left(\frac{3}{8}x^2 - \frac{9}{4}x + 3\right)\right)dx$

$= \int_0^2 \left(-\frac{1}{8}x^2 + \frac{7}{4}x\right)\,dx$

$= \left[-\frac{1}{24}x^3 + \frac{7x^2}{8}\right]_0^2$

$= \left(-\frac{1}{3} + \frac{7}{2}\right) - 0$

$= \frac{19}{6}$ square units

area of plaque $= 2 \times \frac{19}{6}$

$= \frac{19}{3}$

$= 6\frac{1}{3}$ square units.

5 (a) C_1 $x^2 + y^2 + 6x + 10y + 9 = 0$

$2g = 6$ $2f = 10$ $c = 9$

$g = 3$ $f = 5$

centre $(-3, -5)$ radius $= \sqrt{9 + 25 - 9}$
$= 5$

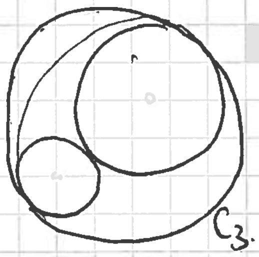
distance between centres $= \sqrt{(9+3)^2 + (11+5)^2}$

$= \sqrt{144 + 256}$

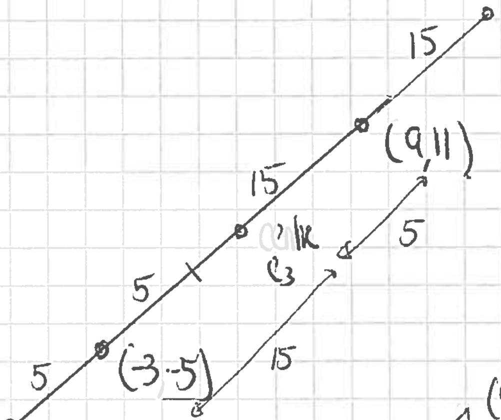
$= 20$

radius of $C_2 = 20 - 5 = 15.$

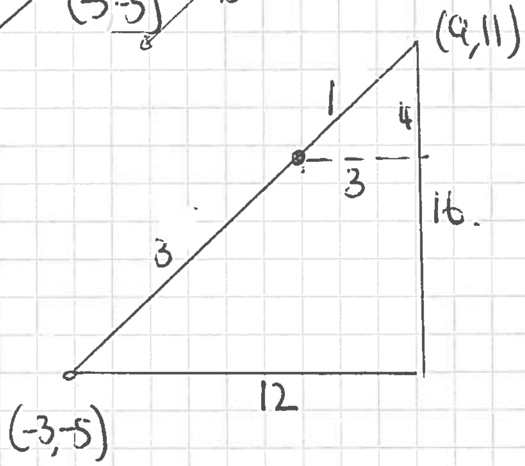
(b)



diameter of $C_3 = 5+5+15+15$
$= 40$
$r = 20$



ratio



centre of C_3
$(9-3,\ 11-4)$
$= (6, 7)$

Equation
$(x-6)^2 + (y-7)^2 = 20^2$
$(x-6)^2 + (y-7)^2 = 400$.

⑥ (a) $\underline{p} \cdot (\underline{q} + \underline{r})$
$= |\underline{p}||\underline{q}| \cos 60 + |\underline{p}||\underline{r}| \cos 90$
$= 3 \times 3 \times \frac{1}{2} + 0$
$= \frac{9}{2}$

(b) $\vec{EC} = \vec{EA} + \vec{AB} + \vec{BC}$
$= -\underline{q} + \underline{p} + \underline{r}$
$= \underline{p} + \underline{r} - \underline{q}$

(c) $\vec{AE} \cdot \vec{EC} = 9\sqrt{3} - \frac{9}{2}$.

$$\vec{AE} \cdot \vec{EC} = \underline{q} \cdot (\underline{p} + \underline{r} - \underline{q})$$
$$= \underline{q} \cdot \underline{p} + \underline{q} \cdot \underline{r} - |\underline{q}|^2$$
$$= |\underline{q}||p| \cos 60 + |\underline{q}||\underline{r}| \cos 30 - |\underline{q}|^2$$
$$= 9 \cdot \frac{1}{2} + 3|\underline{r}| \cdot \frac{\sqrt{3}}{2} - 9$$
$$= 3|\underline{r}| \frac{\sqrt{3}}{2} - \frac{9}{2}$$

Since $\vec{AE} \cdot \vec{EC} = 9\sqrt{3} - \frac{9}{2}$.

$$3|\underline{r}| \frac{\sqrt{3}}{2} - \frac{9}{2} = 9\sqrt{3} - \frac{9}{2}$$
$$3|\underline{r}| \frac{\sqrt{3}}{\cancel{2}} = 9\sqrt{3}.$$
$$|\underline{r}| = \frac{9 \times 2}{3}$$
$$|\underline{r}| = 6.$$

(7)

(a) $\int (3\cos 2x + 1)\, dx$

$= \frac{3}{2}\sin 2x + x + c$

(b) $3\cos 2x + 1 = 4\cos^2 x - 2\sin^2 x$

LHS $= 3\cos 2x + 1$

$= 3(\cos^2 x - \sin^2 x) + 1$

$= 3\cos^2 x - 3\sin^2 x + 1$

$= 3\cos^2 x - 3\sin^2 x + (\cos^2 x + \sin^2 x)$

$= 4\cos^2 x - 2\sin^2 x$. as required.

(c) $\int (\sin^2 x - 2\cos^2 x)\, dx$

$= \int \left(-\frac{1}{2}\right)(4\cos^2 x - 2\sin^2 x)\, dx$

$= -\frac{1}{2}\int 3\cos 2x + 1$

$= -\frac{3}{4}\sin 2x - \frac{1}{2}x + c$.

(8) $h = 36\sin(1.5t) - 15\cos(1.5t) + 65$

$36\sin(1.5t) - 15\cos(1.5t) = k\sin(1.5t - a)$

$= k\sin 1.5t\cos a - k\cos 1.5t\sin a$

$= k\cos a\sin(1.5t) - k\sin a\cos(1.5t)$

$k\cos a = 36$

$k\sin a = 15$

Square and add $k^2 = 36^2 + 15^2$

$k^2 = 1521$

$k = 39$

Divide $\tan\alpha = \frac{15}{36}$

$\alpha = 0.395$ (3 s.f)

all +. radians

So $36\sin(1.5t) - 15\cos(1.5t) = 39\sin(1.5t - 0.395)$

$h = 100$ $\quad 36\sin(1.5t) - 15\sin(1.5t) + 65 = 100$

$39\sin(1.5t - 0.395) = 35$

$\sin(1.5t - 0.395) = \frac{35}{39}$

ra. 1.11

$1.5t - 0.395 = 1.11$ or $\pi - 1.11$

$1.5t - 0.395 = 1.11$ or 2.03

$1.5t = 1.505$ or 2.425

$t = 1.00$ or 1.62

S ✓	A ✓
T	C