X100/12/03

NATIONAL QUALIFICATIONS 2012

MONDAY, 21 MAY
2.50 PM – 4.00 PM

MATHEMATICS
HIGHER
Paper 2

Read Carefully

1 **Calculators may be used in this paper.**

2 Full credit will be given only where the solution contains appropriate working.

3 Answers obtained by readings from scale drawings will not receive any credit.

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

or $\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin 2A &= 2\sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

ALL questions should be attempted. *Marks*

1. Functions f and g are defined on the set of real numbers by

- $f(x) = x^2 + 3$
- $g(x) = x + 4$.

(*a*) Find expressions for:

(i) $f(g(x))$;

(ii) $g(f(x))$. **3**

(*b*) Show that $f(g(x)) + g(f(x)) = 0$ has no real roots. **3**

2. (*a*) Relative to a suitable set of coordinate axes, Diagram 1 shows the line $2x - y + 5 = 0$ intersecting the circle $x^2 + y^2 - 6x - 2y - 30 = 0$ at the points P and Q.

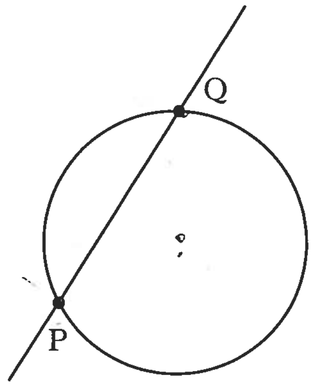


Diagram 1

Find the coordinates of P and Q. **6**

(*b*) Diagram 2 shows the circle from (*a*) and a second congruent circle, which also passes through P and Q.

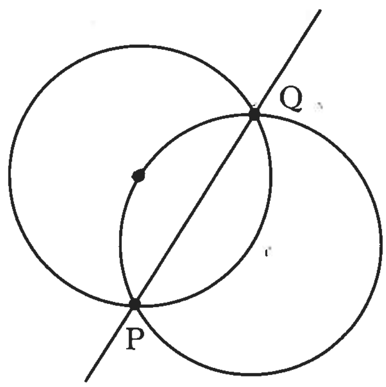
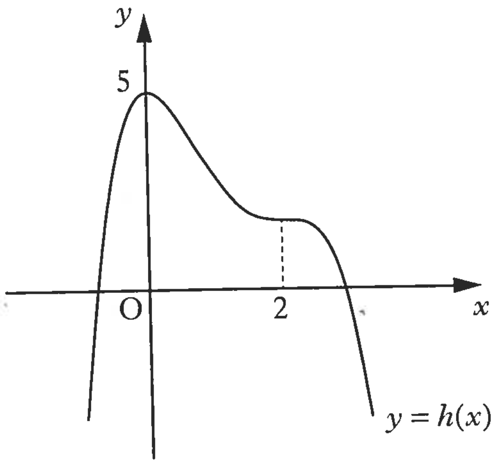


Diagram 2

Determine the equation of this second circle. **6**

3. A function f is defined on the domain $0 \le x \le 3$ by $f(x) = x^3 - 2x^2 - 4x + 6$.
Determine the maximum and minimum values of f. **7**

4. The diagram below shows the graph of a quartic $y = h(x)$, with stationary points at $x = 0$ and $x = 2$.



On separate diagrams sketch the graphs of:

(*a*) $y = h'(x)$; **3**

(*b*) $y = 2 - h'(x)$. **3**

5. A is the point $(3, -3, 0)$, B is $(2, -3, 1)$ and C is $(4, k, 0)$.

(*a*) (i) Express $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in component form.

(ii) Show that $\cos A\hat{B}C = \dfrac{3}{\sqrt{2(k^2 + 6k + 14)}}$. **7**

(*b*) If angle ABC = 30°, find the possible values of k. **5**

Marks

6. For $0 < x < \frac{\pi}{2}$, sequences can be generated using the recurrence relation

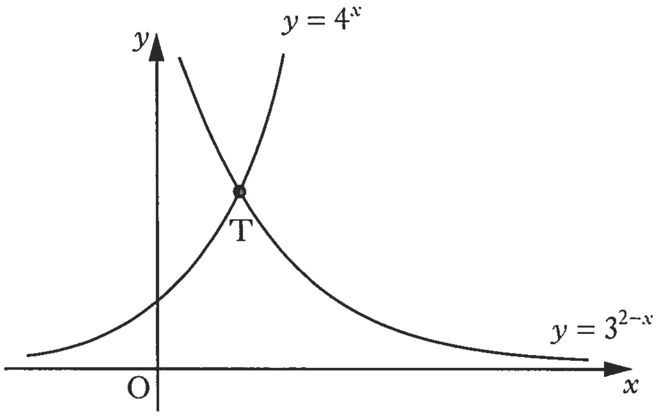
$$u_{n+1} = (\sin x)u_n + \cos 2x, \text{ with } u_0 = 1.$$

(*a*) Why do these sequences have a limit? **2**

(*b*) The limit of one sequence generated by this recurrence relation is $\frac{1}{2}\sin x$.

Find the value(s) of x. **7**

7. The diagram shows the curves with equations $y = 4^x$ and $y = 3^{2-x}$.



The graphs intersect at the point T.

(*a*) Show that the x – coordinate of T can be written in the form $\dfrac{\log_a p}{\log_a q}$, for all $a > 1$. **6**

(*b*) Calculate the y – coordinate of T. **2**

[*END OF QUESTION PAPER*]

Higher Paper 2 2012.

① (a) (i) $f(g(x))$
$= f(x+4)$
$= (x+4)^2 + 3$
$= x^2 + 8x + 16 + 3$
$= x^2 + 8x + 19$

(ii) $g(f(x))$
$= g(x^2+3)$
$= x^2+3+4$
$= x^2+7$

(b) $f(g(x)) + g(f(x))$
$= x^2+8x+19 + x^2+7$
$= 2x^2 + 8x + 26$

$b^2 - 4ac = 64 - 4 \times 2 \times 26$
$= 64 - 208$
$= -144$
< 0 so no real roots.

② Solve $x^2+y^2-6x-2y-30=0$ ... ①
$2x - y + 5 = 0$ ... ②

From ② $y = 2x+5$
In ①

$x^2 + (2x+5)^2 - 6x - 2(2x+5) - 30 = 0$
$x^2 + 4x^2 + 20x + 25 - 6x - 4x - 10 - 30 = 0$
$5x^2 + 10x - 15 = 0$
$x^2 + 2x - 3 = 0$
$(x+3)(x-1) = 0$

$x = -3$ $\qquad$ $x = 1$

$y = -1$ $\qquad$ $y = 7$

$P(-3, -1)$ $\qquad$ $Q(1, 7)$

(b) $\quad x^2 + y^2 - 6x - 2y - 30 = 0$

$2g = -6$ $\qquad$ $2f = -2$ $\qquad$ $c = -30$

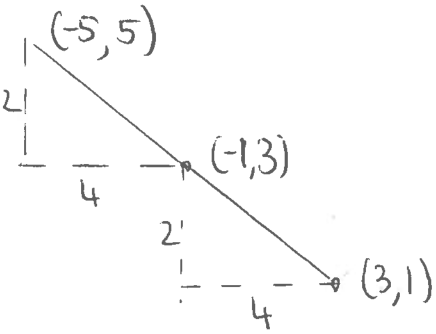
$g = -3$ $\qquad$ $f = -1$

centre $(3, 1)$ $\qquad$ radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{9 + 1 + 30}$

$= \sqrt{40}$

midpoint PQ $(-1, 3)$.



$(x+5)^2 + (y-5)^2 = 40$

(3) $\quad f(x) = x^3 - 2x^2 - 4x + 6$

$f'(x) = 3x^2 - 4x - 4$

For stationary points $\qquad f'(x) = 0$

$3x^2 - 4x - 4 = 0$

$(3x+2)(x-2) = 0$

$x = -\frac{2}{3}$ $\quad$ or $\quad$ $x = 2$

too small $\qquad$ $y = -2$.

Nature.

x	$\underset{(1)}{\rightarrow}$	2	$\underset{(3)}{\rightarrow}$
$f'(x) = (3x+2)(x-2)$	$-$	0	$+$
	\	—	/

$(2,-2)$ is a minimum T.P.

When $x=0$ $\quad f(x) = 6$

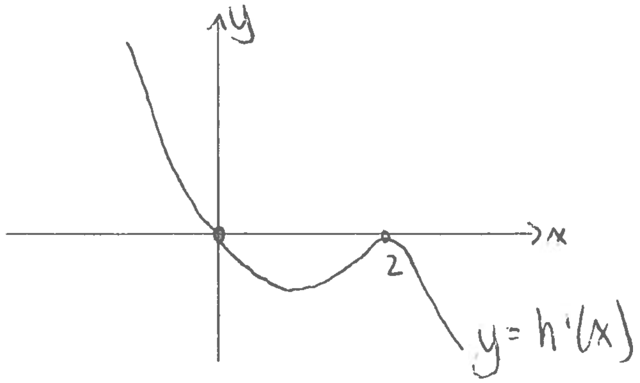
$x=3$ $\quad f(x) = 27-18-12+6$

$= 3$

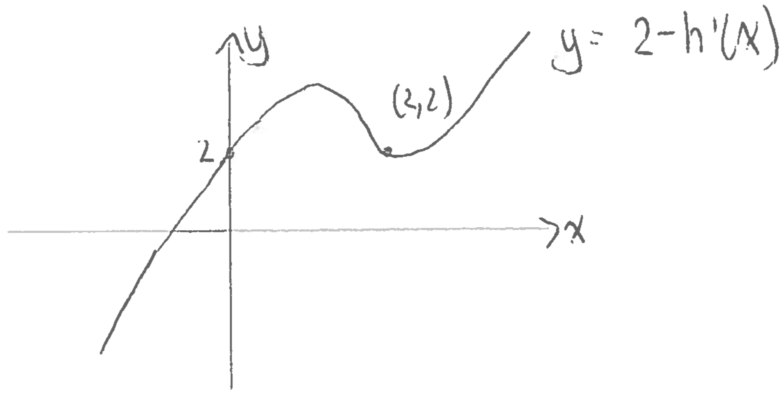
$(0,6)$ is a maximum on $0 < x < 3$

min value -2 max value 6.

(i) (a)



(b)



⑤ (a) (i) $\vec{BA} = \underline{a} - \underline{b}$

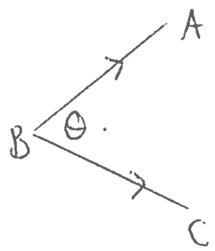
$$= \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\vec{BC} = \underline{c} - \underline{b}$

$$= \begin{pmatrix} 4 \\ k \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ k+3 \\ -1 \end{pmatrix}$$



$$\vec{BA} \cdot \vec{BC} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ k+3 \\ -1 \end{pmatrix}$$

$$= 2 + 1$$

$$= 3$$

$|\vec{BA}| = \sqrt{1+1}$

$= \sqrt{2}$

$|\vec{BC}| = \sqrt{4+(k+3)^2+}$

$= \sqrt{5+k^2+6k}$

$= \sqrt{k^2+6k+14}$

$$\cos\theta = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}||\vec{BC}|}$$

$$= \frac{3}{\sqrt{2}\sqrt{k^2+6k+14}}$$

$$\cos A\hat{B}C = \frac{3}{\sqrt{2(k^2+6k+14)}}$$ as required.

(b) $\cos 30 = \dfrac{3}{\sqrt{2(k^2+6k+14)}}$

$\dfrac{\sqrt{3}}{2} = \dfrac{3}{\sqrt{2(k^2+6k+14)}}$

$\dfrac{3}{4} = \dfrac{9}{2(k^2+6k+14)}$

$6(k^2+6k+14) = 36$

$k^2+6k+14 = 6$

$k^2+6k+8=0$

$(k+4)(k+2)=0$

$\underline{\underline{k=-4}}$ or $\underline{\underline{k=-2}}$

⑥ (a) $|\sin x| < 1$ so limit exists.

(b) $S_\infty = \dfrac{b}{1-a}$

$\dfrac{1}{2}\sin x = \dfrac{\cos 2x}{1-\sin x}$

$\dfrac{1}{2}\sin x(1-\sin x) = \cos 2x$

$\dfrac{1}{2}\sin x - \dfrac{1}{2}\sin^2 x = 1-2\sin^2 x$

$\sin x - \sin^2 x = 2 - 4\sin^2 x$

$3\sin^2 x + \sin x - 2 = 0$

$(3\sin x - 2)(\sin x + 1) = 0$

$\sin x = \dfrac{2}{3}$ or $\sin x = -1$

no solutions for $0 < x < \frac{\pi}{2}$

$x = \sin^{-1} \frac{2}{3}$

$x = 0.730$ (3 s.f)

7 (a) $y = 4^x$ $y = 3^{2-x}$

$$4^x = 3^{2-x}$$

Take $\log_a$

$$\log_a 4^x = \log_a 3^{2-x}$$

$$x \log_a 4 = (2-x) \log_a 3$$

$$x \log_a 4 + x \log_a 3 = 2 \log_a 3$$

$$x (\log_a 4 + \log_a 3) = \log_a 9$$

$$x = \frac{\log_a 9}{\log_a 12}.$$ as required.

When $x = \frac{\log_a 9}{\log_a 12}$

let $a = 10$ $x = 0.884$

$$y = 4^{0.884}$$

$= 3.41$ (3 s.f)