Solving Trig Equations Algebraically

- Check for degrees or radians
- Check limits; how many solutions might you expect ?
- Remember $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$
- Use exact value triangles and graphs where appropriate

Solving Trigonometric Equations in Degrees

Solve

a) $5cosx° + 2 = -2 \quad 0 \leq x \leq 360$

$5\cos x = -4$

$\cos x = -\frac{4}{5}$

r.a $\cos^{-1}\left(\frac{4}{5}\right) = 36.9°$ $\begin{array}{c|c} S\checkmark & A \\ \hline T\checkmark & C \end{array}$

$x = 180 - 36.9°,\ 180 + 36.9°$

$\underline{\underline{x = 143.1°,\ 216.9°}}$

b) $4sint° - 2 = 0 \quad 0 \leq t \leq 540$

bigger than 360

$4\sin t = 2$

$\sin t = \frac{1}{2}$

ra $\sin^{-1}\left(\frac{1}{2}\right) = 30°$ $\begin{array}{c|c} S\checkmark & A\checkmark \\ \hline T & C \end{array}$

$t = 30°,\ 180 - 30°$

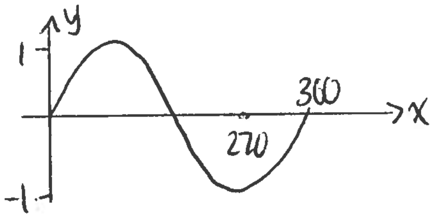
$\underline{\underline{t = 30°,\ 150°}},\ 390°,\ 570°$ ← add 360° to get more answers.

c) $sinx° + 1 = 0 \quad 0 \leq x \leq 360$

$\sin x = -1$

$\underline{\underline{x = 270°}}$

* when trig function = −1, 0 or 1 use graph to find angle



d) $6tanx° - 1 = 4 \quad 0 \leq x \leq 360$

$6\tan x = 5$

$\tan x = \frac{5}{6}$

r.a $\tan^{-1}\left(\frac{5}{6}\right) = 39.8$ $\begin{array}{c|c} S & A\checkmark \\ \hline \checkmark T & C \end{array}$

$x = 39.8°,\ 180 + 39.8$

$\underline{\underline{x = 39.8°,\ 219.8°}}$

p172 Ex 8A

Solving Trigonometric Linear Equations in Degrees with Compound Angles

Solve

a) $3tan2x° = 1$ $0 \le x \le 360$

'2x' so should get 2x2=4 answers up to 360°

ra $\tan^{-1}(\frac{1}{3}) = 18.4°$

S | A✓
✓T | C

$\tan 2x = \frac{1}{3}$

$2x = 18.4,\ 180 + 18.4$

$x = 9.2,\ 99.2$

'2x' ⇒ period 360÷2=180° → add (or subtract) to get more answers $x = 9.2°, 99.2°, 189.2°, 279.2°$

b) $sin(x + 36)° = 0.76$ $0 \le x \le 360$

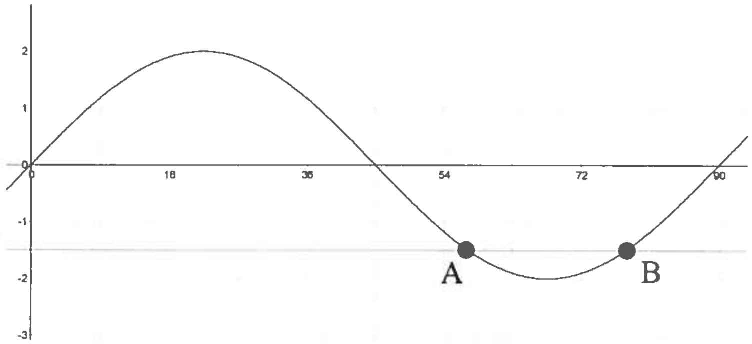
ra $\sin^{-1}(0.76) = 49.5°$

S✓ | A✓
T | C

$x + 36 = 49.5°, 130.5°$

$x = 13.5°, 94.5°$

c) The diagram shows the graph of a sine function from 0° to 90°.



i. State the equation of the graph.

period 90° → 4 waves up to 360°
amplitude 2

$y = 2\sin 4x$

ii. The line with equation $y = -1.5$ intersects the curve at A and B. Find the coordinates of A and B.

Solve simultaneously

$2\sin 4x = -1.5$

ra $\sin^{-1}(0.75) = 48.6°$

$\sin 4x = -0.75$

S | A
✓T | C✓

$4x = 180 + 48.6,\ 360 - 48.6$

$4x = 228.6,\ 311.4$ (1d.p)

$x = 57.15,\ 77.85$

p174 Ex 8B

Solving Quadratic Trigonometric Equations in Degrees

Solve

a) $2cos^2x° = 1$ $\qquad\qquad 0 \leq x \leq 360$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm \frac{1}{\sqrt{2}}$

r.a $45°$

S	A
T	C

$x = 45°, 135°, 225°, 315°$

b) $sin^2(3x - 45)° = {}^1/_2$ $\qquad\qquad 0 \leq x \leq 360$

$\sin(3x - 45) = \pm \frac{1}{\sqrt{2}}$

r.a $45°$

S	A
T	C

$3x - 45 = 45, 135, 225, 315$

$3x = 90, 180, 270, 360$

$x = 30, 60, 90, 120$

'$3x$' so period is $120°$ $\rightarrow$ add on to get all answers up to 360

$x = 30, 60, 90, 120,$
$150, 180, 210, 240$
$270, 300, 330, 360.$

c) $2sin^2x° + 3sinx° + 1 = 0 \qquad 0 \leq x \leq 360$

Factorise

$(2\sin x + 1)(\sin x + 1) = 0$

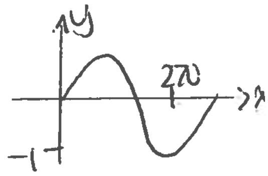
$2\sin x + 1 = 0$ or $\sin x + 1 = 0$

$\sin x = -\frac{1}{2}$ $\qquad$ $\sin x = -1$

ra $30°$

S | A
✓T | C ✓

$x = 210°, 330°$ $\qquad$ $x = 270°$



Solution $\underline{\underline{x = 210°, 270°, 330°}}$

d) $tan^2x° + tanx° = 0 \qquad 0 \leq x \leq 360$

Factorise → common factor

$\tan x(\tan x + 1) = 0$

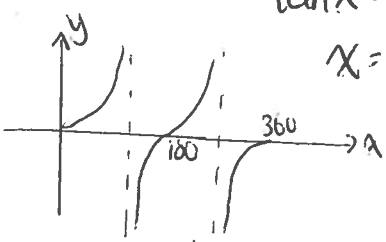
$\tan x = 0$ or $\tan x + 1 = 0$

$x = 0, 180°, 360°$ $\qquad$ $\tan x = -1$

ra $45°$

✓S | A
T | C✓

$x = 135°, 315°$



$y = \tan x$

Solution $\underline{\underline{x = 0, 135, 180, 315, 360}}$

Quadratic

Make $= 0$

Factorise

e) $5cos^2x° = -11cosx° - 2 \qquad 0 \leq x \leq 360$

$5\cos^2 x + 11\cos x + 2 = 0$

$(5\cos x + 1)(\cos x + 2) = 0$

$\cos x = -\frac{1}{5}$ or $\cos x = -2$
not possible.

ra $78.5°$

✓S | A
✓T | C

$\underline{\underline{x = 101.5°, 258.5°}}$

p180 Ex 8C

Solving Linear Trigonometric Equations Involving Compound Angles Using Radians

Solve

a) $2cosx + 1 = 0$ $\qquad 0 \leq x \leq 2\pi$

radians.

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

ra $\frac{\pi}{3}$

S | A
T | C

$x = \pi - \frac{\pi}{3}, \ \pi + \frac{\pi}{3}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

b) $3sinx - 2 = 0$ $\qquad 0 \leq x \leq 2\pi$

radians.

$\sin x = \frac{2}{3}$

ra $\sin^{-1}\left(\frac{2}{3}\right) = 0.730$ (3s.f)

* calculator on radians

S | A
T | C

$x = 0.730, \ \pi - 0.730$

$x = 0.730, \ 2.41$ (3s.f)

c) $1 - \sqrt{2}sin6t = 0$ $\qquad 0 \leq t \leq \pi$

radians.

$-\sqrt{2}\sin 6t = -1$

$\sin 6t = \frac{1}{\sqrt{2}}$

ra $\frac{\pi}{4}$

S | A
T | C

$6t = \frac{\pi}{4}, \ \pi - \frac{\pi}{4}$

$6t = \frac{\pi}{4}, \ \frac{3\pi}{4}$

$t = \frac{\pi}{24}, \ \frac{3\pi}{24}$

'6t' → period $\frac{2\pi}{6} = \frac{\pi}{3} = \frac{8\pi}{24}$ → add on to get all answers up to π

$t = \frac{\pi}{24}, \frac{3\pi}{24}, \frac{9\pi}{24}, \frac{11\pi}{24}, \frac{17\pi}{24}, \frac{19\pi}{24}$

p182 Ex 8D

$t = \frac{\pi}{24}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{11\pi}{24}, \frac{17\pi}{24}, \frac{19\pi}{24}$

Solving Quadratic Trigonometric Equations Using Radians

Solve

a) $2cos^2x - 1 = 0 \qquad 0 \leq x \leq \pi$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\frac{1}{\sqrt{2}} \qquad ra\ \frac{\pi}{4}$

S | A
T | C

$x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

b) $4sin^2(\theta - \pi/_2) - 3 = 0 \qquad 0 \leq \theta \leq 2\pi$

$\sin^2(\theta - \frac{\pi}{2}) = \frac{3}{4}$

$\sin(\theta - \frac{\pi}{2}) = \pm\frac{\sqrt{3}}{2} \qquad ra\ \frac{\pi}{3}$

S | A
T | C

$\theta - \frac{\pi}{2} = \frac{\pi}{3}, \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$\theta - \frac{\pi}{2} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\theta = \frac{\pi}{3} + \frac{\pi}{2}, \frac{2\pi}{3} + \frac{\pi}{2}, \frac{4\pi}{3} + \frac{\pi}{2}, \frac{5\pi}{3} + \frac{\pi}{2}$

$\theta = \frac{2\pi}{6} + \frac{3\pi}{6}, \frac{4\pi}{6} + \frac{3\pi}{6}, \frac{8\pi}{6} + \frac{3\pi}{6}, \frac{10\pi}{6} + \frac{3\pi}{6}$

$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$

c) $4sin^2z + 4sinz - 3 = 0 \qquad 0 \le z \le \pi$

Quadratic → factorise

$(2\sin z + 3)(2\sin z - 1) = 0$

$\sin z = -\frac{3}{2}$ not possible

$\sin z = \frac{1}{2}$

ra $\frac{\pi}{6}$

S | A
T | C

$z = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$z = \frac{\pi}{6}, \frac{5\pi}{6}$

p186 Ex 8E

Solving Trigonometric Equations in Degrees Using Identities

Solve

a) $5sin^2x° - 2cosx° - 2 = 0 \quad 0 \leq x \leq 360$ ← degrees.

$\sin^2 x = 1 - \cos^2 x$ (substitute so equation only contains cosx)

$5(1-\cos^2 x) - 2\cos x - 2 = 0$

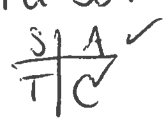
$5 - 5\cos^2 x - 2\cos x - 2 = 0$

$5\cos^2 x + 2\cos x - 3 = 0$

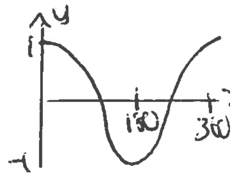
$(5\cos x - 3)(\cos x + 1) = 0$

$\cos x = \frac{3}{5}$ or $\cos x = -1$

ra 53.1°



$x = 53.1°, 306.9°$ $\qquad x = 180°$



Solution $x = 53.1°, 180°, 306.9°$

b) $sin2x° - cosx° = 0 \quad 0 \leq x \leq 360$ ← degrees

(double angle use formula $\sin 2x = 2\sin x\cos x$)

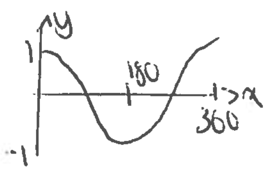
$2\sin x\cos x - \cos x = 0$

Factorise

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ $\qquad \sin x = \frac{1}{2}$ $\qquad$ ra 30°

$x = 90°, 270°$ $\qquad x = 30°, 150°$



Solution $x = 30°, 90°, 150°, 270°$

c) $2cos2x° = 2 - 6cosx°$ $\quad 0 \leq x \leq 360$

double angle → use formula containing only cosx since equation already contains cosx

$\cos 2x = 2\cos^2 x - 1$

$2\cos 2x = 2 - 6\cos x$

$2(2\cos^2 x - 1) = 2 - 6\cos x$

$2\cos^2 x - 1 = 1 - 3\cos x$

$2\cos^2 x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -2$ not possible

ra $60°$

S	A ✓
T	C ✓

$x = 60°, 300°$

Solution $x = 60°, 300°$

p191 Ex 8F

Solving Trigonometric Equations in Radians Using Identities

Solve

$cos2x = -3cosx - 2$ $\qquad$ $0 \leq x \leq 2\pi$ $\quad$ radians

double angle.
equation contains cosx $\rightarrow$ use $\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x - 1 = -3\cos x - 2$

$2\cos^2 x + 3\cos x + 1 = 0$

$(2\cos x + 1)(\cos x + 1) = 0$

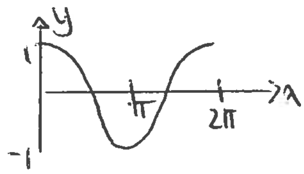
$\cos x = -\frac{1}{2}$ $\qquad$ $\cos x = -1$

ra $\frac{\pi}{3}$

✓S | A
✓T | C

$x = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$ $\qquad$ $x = \pi$

$x = \frac{2\pi}{3},\ \frac{4\pi}{3}$ $\qquad$ $x = \pi$



Solution $x = \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3}$

p193 Ex 8G

Solving Trigonometric Equations Using The Wave Function

Solve ← (cos and sin function <u>same</u> angle → use wave function)

a) $\sqrt{3}cos\theta + sin\theta = \sqrt{2}$ $\qquad 0 \le \theta \le 2\pi$

↑ radians.

Write. $\sqrt{3}\cos\theta + \sin\theta$ in the form $k\cos(\theta - \alpha)$

$$\sqrt{3}\cos\theta + \sin\theta = k\cos(\theta - \alpha)$$
$$= k\cos\theta\cos\alpha + k\sin\theta\sin\alpha.$$
$$= (k\cos\alpha)\cos\theta + (k\sin\alpha)\sin\theta.$$

Compare $\qquad \sqrt{3} \quad \cos\theta \quad + \quad \sin\theta.$

So $\qquad k\cos\alpha = \sqrt{3}$
$\qquad\quad k\sin\alpha = 1$

Square and add $\qquad k^2 = 3^2 + 1$
$\qquad\qquad\qquad\qquad k = 2$

Divide $\qquad \tan\alpha = \frac{1}{\sqrt{3}}$

$$\alpha = \frac{\pi}{6}$$

$$\sqrt{3}\cos\theta + \sin\theta = 2\cos\left(\theta - \frac{\pi}{6}\right)$$

Solve $\qquad 2\cos\left(\theta - \frac{\pi}{6}\right) = \sqrt{2}$

$$\cos\left(\theta - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$$
$$= \frac{1}{\sqrt{2}}$$

ra $\frac{\pi}{4}$

S	A ✓
T	C ✓

$$\theta - \frac{\pi}{6} = \frac{\pi}{4}, \frac{7\pi}{4}$$
$$\theta = \frac{5\pi}{12}, \frac{23\pi}{12}$$

b) $2cos2x° - 3sin2x° = 1$ $\quad 0 \leq x \leq 360$

$2\cos 2x - 3\sin 2x = k\cos(2x - \alpha)$

$= k\cos 2x \cos\alpha + k\sin 2x \sin\alpha$

$= (k\cos\alpha)\cos 2x + (k\sin\alpha)\sin 2x$

Compare $\quad 2 \quad \cos 2x \quad - \quad 3 \quad \sin 2x$

So $\quad k\cos\alpha = 2$

$k\sin\alpha = -3$

Square and add $\quad k^2 = 4+9$

$k = \sqrt{13}.$

Divide $\quad \tan\alpha = -\frac{3}{2}$

$\alpha = 360 - 56.3$

$\alpha = 303.7°$

ra $\tan^{-1}\frac{3}{2}$

$= 56.3°$

S | A
T | C ✓

cos +
sin −
tan −

So $\quad 2\cos 2x - 3\sin 2x = \sqrt{13}\cos(2x - 303.7°)$

Solve $\quad 2\cos 2x - 3\sin 2x = 1$

$\sqrt{13}\cos(2x - 303.7) = \frac{1}{\sqrt{13}}$

ra 73.9°

S | A ✓
T | C ✓

$2x - 303.7 = 73.9, \; 360 - 73.9$

$2x = 73.9 + 303.7, \; 286.1 + 303.7$

$x = \frac{377.6}{2}, \; \frac{589.8}{2}$

$x = 188.8°, \; 294.9°$

period $\frac{360}{2} = 180°$

so $\underline{\underline{x = 8.8°, 114.9°, 188.8°, 294.9°}}$

(subtracting 180° from other 2 answers.)

c) The formula $d(t) = 600 + 100(cos30t° - sin30t°)$ gives an approximation to the depth, d, in centimetres, of water in a harbour t hours after midnight.

i. Express $f(t) = cos30t° - sin30t°$ in the form $kcos(30t - \alpha)°$ and state the values of k and α, where $0 \leq \alpha \leq 360$.

$\cos 30t - \sin 30t = k\cos(30t - \alpha)$
$= k(\cos 30t \cos\alpha - \sin 30t \sin\alpha)$
$= k\cos\alpha \cos 30t - k\sin\alpha \sin 30t$

$k\cos\alpha = 1$
$k\sin\alpha = 1$

Square and add $\quad k^2 = 2$
$k = \sqrt{2}$

Divide $\quad \tan\alpha = 1$
$\alpha = 45°$ $\quad$ S | A ✓ / T | C

So $f(t) = \cos 30t - \sin 30t = \sqrt{2}\cos(30t - 45°)$

ii. Use your results from (i) to help you sketch the graph of $f(t)$ for $0 \leq t \leq 12$.

period $\frac{360}{30} = 12$.

max $\sqrt{2}$ when angle = 0, 360.
$30t - 45 = 0, 360$
$30t = 45, 405$
$t = 1.5, 13.5$ (too big)

min $-\sqrt{2}$ when angle = 180°
ie $30t - 45 = 180$
$30t = 225$
$t = 7.5$

iii. Hence, on a separate diagram, sketch the graph of $d(t)$ for $0 \leq t \leq 12$. $\quad$ * see graph paper.

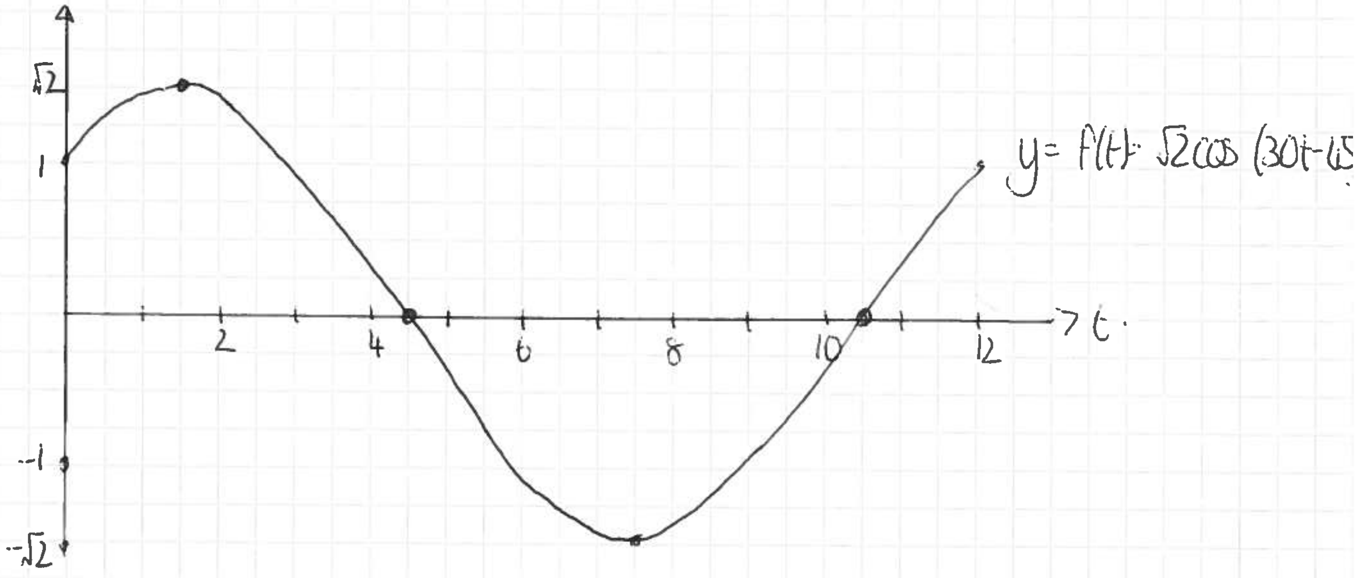
$d(t) = 600 + 100(\cos 30t - \sin 30t)$
$= 600 + 100\sqrt{2}\cos(30t - 45)$

From f(t) $\quad$ add 600 (ie move up) $\quad$ (multiply y co-ords by 100)

* see graph paper.

iv. When is the first low water time at the harbour?

minimum angle = 180°
$\Rightarrow t = 7.5$
ie $7\frac{1}{2}$ hours after midnight
0730.



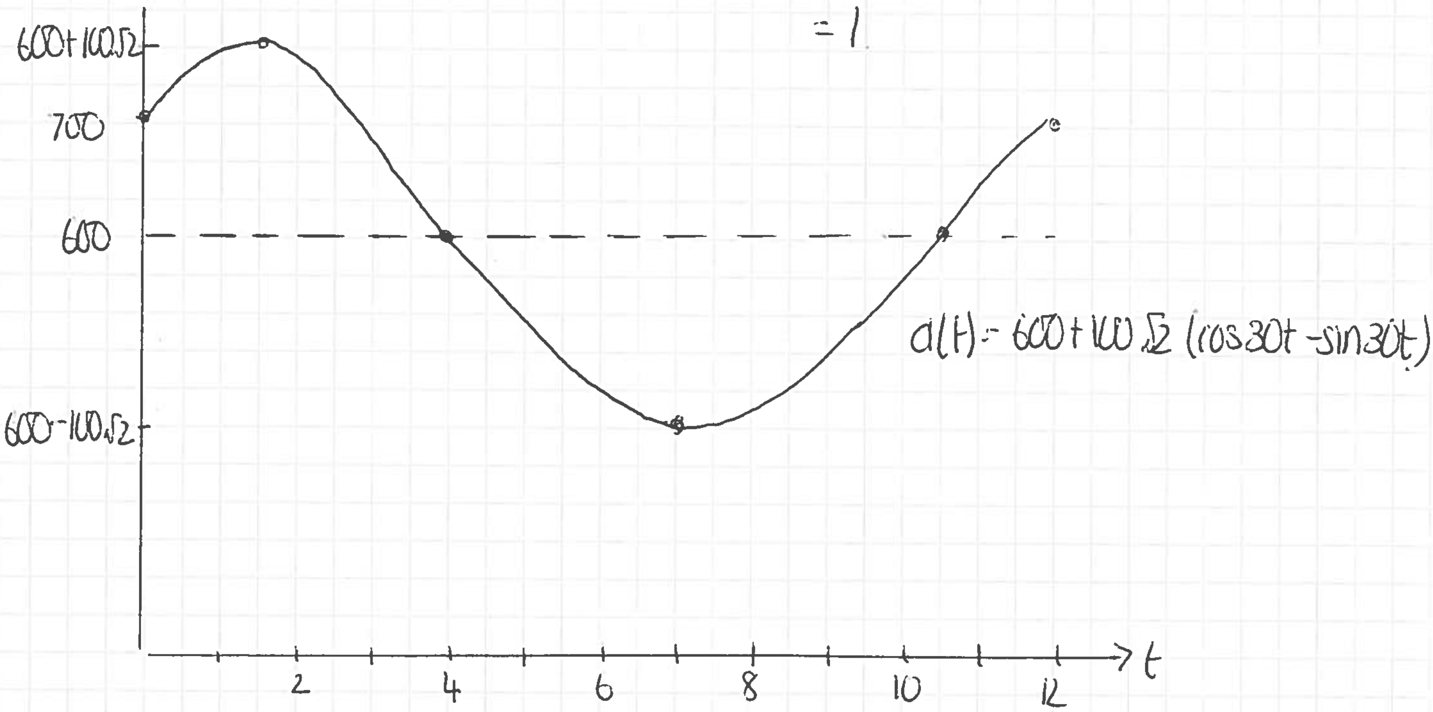
(c) (ii) cont.

cuts t-axis $\quad f(t) = 0$

$$\sqrt{2}\cos(30t-45) = 0$$
$$30t - 45 = 90,\ 270$$
$$30t = 135,\ 315$$
$$t = 4.5,\ 10.5$$

cuts y-axis $\quad t = 0$

$$f(t) = \sqrt{2}\cos(-45)$$
$$= \sqrt{2} \times \frac{1}{\sqrt{2}}$$
$$= 1$$



v. The local fishing fleet needs at least 5 metres depth of water to enter the harbour without risk of running aground. Between what times must it avoid entering the harbour?

$$600 + 100(\cos 30t - \sin 30t) = 500$$
$$600 + 100\sqrt{2}\cos(30t - 45) = 500$$
$$100\sqrt{2}\cos(30t - 45) = -100$$
$$\cos(30t - 45) = \frac{-1}{\sqrt{2}}$$
$$30t - 45 = 135^\circ, 225$$
$$30t = 180, 270$$
$$t = 6, 9$$

S✓ | A
T✓ | C
ra 45°

The boat should not enter the harbour between 6am and 9am.

p198 Ex 8H

Further Trigonometric Equations

Solve

a) $cosx°cos50° - sinx°sin50° = 0.5$ $\quad 0 \leq x \leq 360$

spot pattern for cos (A+B)

$\cos(x+50) = 0.5$

$x+50 = 60°, 300°$

$x = 10°, 250°$

ra $60°$

S | A ✓
T | C ✓

b)

i. Show that
$$sin(x + {}^{\pi}/_{3}) + sin(x - {}^{\pi}/_{3}) = sinx$$

$LHS = \sin(x+\frac{\pi}{3}) + \sin(x-\frac{\pi}{3})$

$= \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} + \sin x\cos\frac{\pi}{3} - \cos x\sin\frac{\pi}{3}$

$= 2\sin x\cos\frac{\pi}{3}$

$= 2\sin x \times \frac{1}{2}$

$= \sin x$

$=$ RHS as required.

ii. Hence solve the equation
$$sin(x + {}^{\pi}/_{3}) + sin(x - {}^{\pi}/_{3}) = 0.5 \quad 0 \leq x \leq 2\pi$$

$\Rightarrow \sin x = 0.5$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

ra $\frac{\pi}{6}$

S ✓ | A ✓
T | C

p200 Ex 8I