VECTORS

Recall

1) Position vector of a point P(x,y,z) is $\overrightarrow{OP} = \underline{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

3) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

4) $\underline{a} + \underline{o} = \underline{o} + \underline{a} = \underline{a}$ $\qquad \underline{o}$ zero vector = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5) $\overrightarrow{AB} = \underline{b} - \underline{a}$

6) Negative of a vector $\quad \underline{a} + (-\underline{a}) = \underline{o} \qquad \underline{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad -\underline{a} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

7) Multiplication by a scalar

for $\underline{a}$ non-zero vector, **k** non-zero number

(i) $|\mathbf{k}\underline{a}| = \mathbf{k}|\underline{a}|$
(ii) If $\mathbf{k} > 0$, $\mathbf{k}\underline{a}$ is parallel to $\underline{a}$, same sense
(iii) If $\mathbf{k} < 0$, $\mathbf{k}\underline{a}$ is parallel to $\underline{a}$, opposite sense

8) <u>Unit Vectors</u> – vectors with magnitude 1

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and P (x,y,z) then $\underline{p} = x\underline{i} + y\underline{j} + z\underline{k}$

The unit vector in the direction of vector p is given by $\dfrac{\underline{p}}{|\underline{p}|}$

9) Magnitude of a vector

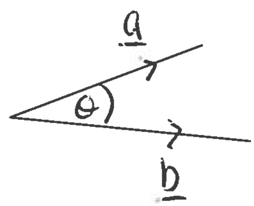
$$\underline{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{then } |\,\underline{a}\,| = \sqrt{(x^2 + y^2 + z^2)}$$

A (x_A , y_A , z_A) and B (x_B , y_B , z_B)

$$|\,\overrightarrow{AB}\,| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \qquad (\overrightarrow{AB} = \underline{b} - \underline{a})$$

10) Scalar (Dot) Product

$$\underline{a}.\underline{b} = |\,\underline{a}\,|\,|\,\underline{b}\,| \cos\theta$$



$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{a}.\underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

$$\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$$

Maths in Action Book 3 Page 44 Exercise 1 Questions 1 , 3 , 4 , 7 , 8 , 10 , 11

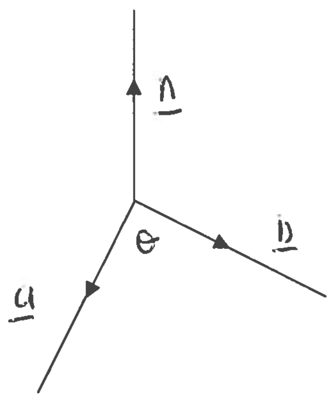
CR p 281 Exercise 15.1

Vector Product (Cross Product)

The vector product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is a vector denoted by $\underset{\sim}{a} \times \underset{\sim}{b}$ ($\underset{\sim}{a}$ cross $\underset{\sim}{b}$) and defined by

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta\ \underset{\sim}{n}$$

where θ is the angle between the positive directions of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\underset{\sim}{n}$ is the unit vector perpendicular to the plane defined by $\underset{\sim}{a}$ and $\underset{\sim}{b}$ ($\underset{\sim}{n}$ is a normal to the plane)



a x b gives a vector perpendicular to a and b

↳ important for finding equation of a plane. (see later)

$\underset{\sim}{a} \times \underset{\sim}{b}$ has magnitude $|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$

has direction perpendicular to (⊥) $\underset{\sim}{a}$ and $\underset{\sim}{b}$

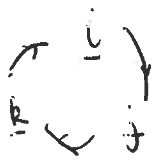
has sense determined by 'right hand rule'

i.e. on right hand hold thumb ($\underset{\sim}{n}$) perpendicular to fore finger ($\underset{\sim}{a}$) and middle finger ($\underset{\sim}{b}$). Rotation of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$ results in an anti-clockwise rotation of $\underset{\sim}{n}$

Rules for Vector Product

- $\underline{a} \times \underline{b}$ **is a vector perpendicular to $\underline{a}$ and $\underline{b}$**
- $\underline{a} \times \underline{b} = \underline{o} \Leftrightarrow \underline{a}$ is parallel to $\underline{b}$ or either $\underline{a} = \underline{o}$ or $\underline{b} = \underline{o}$

  ↳ since $\sin\theta = 0 \Rightarrow \theta = 0$ ie angle between $\underline{a}$ and $\underline{b}$ is zero
- $\underline{a} \times \underline{a} = \underline{o}$



- $\underline{i} \times \underline{j} = \underline{k}$ $\quad \underline{j} \times \underline{k} = \underline{i}$ $\quad \underline{k} \times \underline{i} = \underline{j}$

  (clockwise)
- $\underline{j} \times \underline{i} = -\underline{k}$ $\quad \underline{k} \times \underline{j} = -\underline{i}$ $\quad \underline{i} \times \underline{k} = -\underline{j}$

  (anti-clockwise so negative sense)
- $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$
- $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$
- $\mathbf{k}\underline{a} \times \underline{b} = \mathbf{k}(\underline{a} \times \underline{b})$
- $\mathbf{k}\underline{a} \times \mathbf{l}\underline{b} = \mathbf{kl}(\underline{a} \times \underline{b})$

  ↑ not in bold corrected in pupils notes

Vector Product in Component Form

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then

For easy evaluation use determinant of 3 x 3 matrix

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

← first vector
← second vector

* * learn.

e.g. $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{vmatrix}$ ← first vector ← second vector.

$$= \underline{i}\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= \underline{i}\,(9-2) - \underline{j}\,(6-(-1)) + \underline{k}\,(-4-3)$$

$$= 7\underline{i} - 7\underline{j} - 7\underline{k}$$

<u>Examples</u>

1) P (0 , 5 , 1) Q (3 , 2 , 1) R (-2 , 5 , -3)

Calculate $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$\overrightarrow{PQ} = \underline{q} - \underline{p} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PR} = \underline{r} - \underline{p} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -3 & 0 \\ -2 & 0 & -4 \end{vmatrix}$$

$$= \underline{i}\,(12-0) - \underline{j}\,(-12-0) + \underline{k}\,(0-6)$$

$$= 12\underline{i} + 12\underline{j} - 6\underline{k}$$

2) Find a unit vector perpendicular to both $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

vector perpendicular to $\underline{a}$ and $\underline{b}$ is $\underline{a} \times \underline{b}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \underline{i}(2-1) - \underline{j}(4+1) + \underline{k}(-2,-1)$$

$$= \underline{i} - 5\underline{j} - 3\underline{k}$$

$$|\underline{a} + \underline{b}| = \sqrt{1+25+9}$$

$$= \sqrt{35}$$

unit vector $= \frac{1}{\sqrt{35}}\underline{i} - \frac{5}{\sqrt{35}}\underline{j} - \frac{3}{\sqrt{35}}\underline{k}$

Tips for success.....

- to find a vector perpendicular to two others find the cross product.
- to get a unit vector, find the magnitude of given vector and divide by it.

Maths in Action Book 3 Page 52 Exercise 4 Questions 1(a), (b) , 2(a), (b) , 3 , 5 , 6(a), 8(a), 14(a)

CfE p 286 Ex 15.3

Scalar Triple Product

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors then $\underline{a} \,.\, (\underline{b} \times \underline{c})$ is a scalar and is called a **scalar triple product**

It can be shown that

$$\underline{a} \,.\, (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Examples

Given $\underline{a} = \underline{i} - \underline{j} + \underline{k}$

$\underline{b} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

$\underline{c} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\underline{d} = -2\underline{i} + \underline{j} + 3\underline{k}$

Verify that $\underline{a}.(\underline{b} \times (\underline{c} + \underline{d})) = \underline{a}.(\underline{b} \times \underline{c}) + \underline{a}.(\underline{b} \times \underline{d})$

LHS $\underline{c} + \underline{d} = \underline{i} - \underline{j} + 4\underline{k}$

so LHS $= \underline{a} \,.\, (\underline{b} \times (\underline{c} + \underline{d}))$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 4 \end{vmatrix}$$

$= 1(12+4) + 1(8-4) + 1(-2-3)$

$= 16 + 4 - 5$

$= 15$

RHS $\quad \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 3 & -2 & 1 \end{vmatrix}$

$= 1(3+8) + 1(2-12) + 1(-4-9)$

$= 11 - 10 - 13$

$= -12$

$\underline{a} \cdot (\underline{b} \times \underline{d}) = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ -2 & 1 & 3 \end{vmatrix}$

$= 1(9-4) + 1(6+8) + 1(2+6)$

$= 5 + 14 + 8$

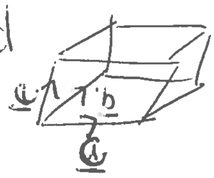
$= 27$

So $\quad$ RHS $= -12 + 27$

$= 15$

$=$ LHS $\qquad$ Hence result.

NB $\quad \underline{a} \cdot (\underline{b} \times \underline{c})$ gives the volume of a parallelopiped

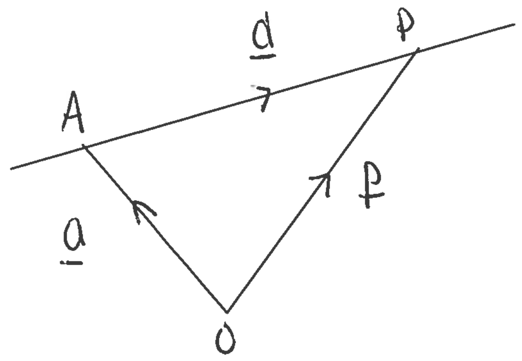


Note

- $\underline{a} \cdot (\underline{b} \times \underline{c})$ is a number
- the dot and cross are interchangeable $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$
- If any of $\underline{a}$, $\underline{b}$ or $\underline{c}$ is zero then $\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$
- If any two of $\underline{a}$, $\underline{b}$ or $\underline{c}$ are parallel then $\underline{a} \cdot (\underline{b} \times \underline{c})$ is 0.

Maths in Action Book 3 Page 55 Exercise 5 Questions 1, 3 , 4 , 7 $\qquad$ CFe p 288 Ex 15.1

The Equation Of A Line



To find the equation of a line in 3D we need
- a point on the line
- its direction vector

instead of gradient.

A straight line in direction of vector $\underline{d}$ passing through the point A with position vector $\underline{a}$

There are three ways we can write the equation of a line

a) If P is any point on the line then

$$\underline{p} = \underline{a} + t\underline{d} \qquad \text{(t a scalar)}$$

This is called the **vector equation** of the line.

b) If A is point (a_1, a_2, a_3) and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and P is any point (x,y,z) then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$x = a_1 + td_1$$
$$y = a_2 + td_2$$
$$z = a_3 + td_3$$

These are called the **parametric equations** of the line.

c) If we eliminate the parameter t we get

point on top

$$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3} \quad (=t)$$

direction vector on bottom.

This is called the equation of the line in **symmetrical form**.

Note If one of the components of the direction vector is zero we do not write the line in symmetric form – use parametric equations instead.

Examples

1) Find the equation of the line joining A(1,0,2) and B(2,1,0) in all three forms.

direction of line $\underline{d} = \vec{AB}$

$= \underline{b} - \underline{a}$

$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Using point A(1,0,2)

$\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

vector form $\underline{p} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

parametric form $x = 1+t$

$y = t$

$z = 2-2t$

symmetric form

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-2}{-2} \quad (=t)$$

2) Find the symmetrical form of the equation through the point (6,3,-5)

(i) in direction $\begin{pmatrix} 4 \\ -8 \\ 7 \end{pmatrix}$

(ii) parallel to line $\dfrac{x}{3} = \dfrac{y-10}{-2} = \dfrac{z+8}{13}$

(i) point $(6, 3, -5)$
direction $\begin{pmatrix} 4 \\ -8 \\ 7 \end{pmatrix}$

$$\frac{x-6}{4} = \frac{y-3}{-8} = \frac{z+5}{7} \quad (=t)$$

(ii) $\dfrac{x}{3} = \dfrac{y-10}{-2} = \dfrac{z+8}{13}$.
direction vector $\begin{pmatrix} 3 \\ -2 \\ 13 \end{pmatrix}$
point $(6, 3, -5)$
equation $\dfrac{x-6}{3} = \dfrac{y-3}{-2} = \dfrac{z+5}{13}$.

Tips for success.....

- Since any point on the line can be used there are an infinite number of ways of writing the equation.
  eg in example ① if we used point B instead equation is $\dfrac{x-2}{1} = \dfrac{y-1}{1} = \dfrac{z}{-2}$.
- also the bottom numbers (direction vector) could be a scalar multiple of those given.

Maths in Action Book 3 Page 52 Exercise 9A Questions 1(a), (b) , 2(a), (b) , 3 (a), (c), (e) , 6

- if direction vector has a zero component use parametric form and not symmetric form.

CfE
p 248 Ex15.?
①(a)
② (a)
③ (a)
⑤
Ex 15.9
②

13
11

The Equation of a Plane

Suppose a point P with position vector $\underline{r}$ lies in the plane Π which passes through fixed point A, with position vector $\underline{a}$, and that $\underline{u}$ and $\underline{v}$ are two non-parallel vectors in the plane then

point — two vectors on plane

$$\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v} \qquad \text{(where } \lambda, \mu \text{ are scalars)}$$

This is the **vector equation** of a plane

If $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ then

$$x = a_1 + \lambda u_1 + \mu v_1$$
$$y = a_2 + \lambda u_2 + \mu v_2$$
$$z = a_3 + \lambda u_3 + \mu v_3$$

These are the **parametric equations** of the plane

If vector $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a normal to the plane i.e. perpendicular to every vector which lies in the plane, then

$$\underline{n}.\overrightarrow{AP} = 0$$
$$\underline{n}.(\underline{r} - \underline{a}) = 0$$

Use to find the equation of a plane. ** $\boxed{\underline{n}.\underline{r} = \underline{n}.\underline{a}}$ $\underline{a}$ = point on plane, $\underline{n}$ = normal vector

Since $\underline{n}.\underline{r} = ax + by + cz$ and $\underline{n}.\underline{a} = aa_1 + ba_2 + ca_3 = k$ it gives rise to

ax + by + cz = k

This is called the **Cartesian Equation** of the plane.

usually use this form

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is normal vector

14 12

Advanced Higher Maths : Unit 3
1.2 Applying Algebraic and Geometric Skills to Vectors

A plane in space can be uniquely defined if

1. Three points on the plane are known
2. Two lines on the plane are known
3. One point on the plane and a normal to the plane are known
4.

Example

1) Find the equation of the plane passing through A(3,-2,0) B(2,0,3) and C(1,-1,1) in

   (i) vector form
   (ii) parametric form
   (iii) Cartesian form

(i) $\underline{a} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Find two vectors in plane eg $\underline{u} = \vec{AB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v} = \vec{AC} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Since $\underline{u}$ and $\underline{v}$ are non-parallel

$\underline{r} = \underline{a} + \lambda \underline{u} + \mu \underline{v}$

$\underline{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ ✱

(ii) Since $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from ✱

$x = 3 - \lambda - 2\mu$
$y = -2 + 2\lambda + \mu$
$z = 3\lambda + \mu$

✱ ✱✱ (iii) $\underline{n}.\underline{r} = \underline{n}.\underline{a}$

Find normal vector ie $\vec{AB} \times \vec{AC}$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2 & 3 \\ -2 & 1 & 1 \end{vmatrix} = \underline{i}(2-3) - \underline{j}(-1+6) + \underline{k}(-1+4)$

$= -\underline{i} - 5\underline{j} + 3\underline{k}$

Plane $\underline{n}.\underline{r} = \underline{n}.\underline{a}$

$\begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

$-x - 5y + 3z = -3 + 10 + 0$

$x + 5y - 3z = -7$

1.2 Applying Algebraic and Geometric Skills to Vectors

2) Find the Cartesian equation of the plane through (-1,2,3) containing direction vectors $8\underline{i}+5\underline{j}+\underline{k}$ and $-4\underline{i}+5\underline{j}+7\underline{k}$

let $\underline{u} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 8 & 5 & 1 \\ -4 & 5 & 7 \end{vmatrix}$$

$$= \underline{i}\,(35-5) - \underline{j}\,(56+4) + \underline{k}\,(40+20)$$

$$= 30\underline{i} - 60\underline{j} + 60\underline{k}$$

$$= 30(\underline{i} - 2\underline{j} + 2\underline{k})$$

Take $\underline{n} = \underline{i} - 2\underline{j} + 2\underline{k}$

Equation $\underline{n}.\underline{r} = \underline{n}.\underline{a}$

$$\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$x - 2y + 2z = 1.$$

cfe p291 Ex 15.5 ① (a)(b) ② (a) ③ (a)
④ (a) ⑤ (b)

3) Find the Cartesian equation of the plane containing P(3, -2, -7) and the line $\frac{x-5}{3}=\frac{y}{1}=\frac{z+6}{4}$

P(3, -2, -7) is on plane.
Q (5, 0, -6) is on line and hence on plane.
so $\overrightarrow{PQ}$ is a vector on plane.

$$\overrightarrow{PQ} = \begin{pmatrix}5\\0\\-6\end{pmatrix} - \begin{pmatrix}3\\-2\\-7\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

direction vector of line (also on plane) $= \begin{pmatrix}3\\1\\4\end{pmatrix}$

Normal vector: $\begin{pmatrix}2\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 2 & 1\\ 3 & 1 & 4\end{vmatrix}$

$$= \underline{i}(8-1) - \underline{j}(8-3) + \underline{k}(2-6)$$

$$= 7\underline{i} - 5\underline{j} - 4\underline{k}$$

Equation $\underline{n}.\underline{r} = \underline{n}.\underline{a}$

$$\begin{pmatrix}7\\-5\\-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}7\\-5\\-4\end{pmatrix}.\begin{pmatrix}3\\-2\\-7\end{pmatrix}$$

$$7x - 5y - 4z = 21 + 10 + 28$$

$$7x - 5y - 4z = 59$$

4) Find the equation of the plane containing the lines $\frac{x-3}{2}=\frac{y+1}{3}=\frac{z}{4}$ and $\frac{x-2}{3}=\frac{y}{2}=\frac{z-2}{2}$

$(3,-1,0)$ and $(2,0,2)$ are on the plane.

direction vectors $\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}3\\2\\2\end{pmatrix}$ are non parallel vectors in the plane.

normal $\begin{pmatrix}2\\3\\4\end{pmatrix} \times \begin{pmatrix}3\\2\\2\end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 4 \\ 3 & 2 & 2 \end{vmatrix}$

$= \underline{i}(6-8) - \underline{j}(4-12) + \underline{k}(4-9)$

$= -2\underline{i} + 8\underline{j} - 5\underline{k}$

equation $\underline{n}.\underline{r} = \underline{n}.\underline{a}$

$\begin{pmatrix}-2\\8\\-5\end{pmatrix} . \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2\\8\\-5\end{pmatrix} \begin{pmatrix}3\\-1\\0\end{pmatrix}$

$-2x + 8y - 5z = -6 - 8 - 0$

$2x - 8y + 5z = 14$

Tips for success.....

To find equation of plane use

$\underline{n}.\underline{r} = \underline{n}.\underline{a}$

need → normal vector (usually use cross product)

→ point on plane.

Maths in Action Book 3 Page 57 Exercise 6 Questions 1(a), (b) , 2(a), (b) , 4(a), (b) , 5 (a), (b)

(CfE p291 Ex 15.5 select)
p302 Ex 15.11 ① (iv)

Finding the Angle Between Two Lines

The angle **between two lines** is the angle between their direction vectors and can be found using the **scalar product**.

Example

Find the size of the angle between the lines

x-1 = y = z-1 and **x=1 + t, y = 5t, z = -t**

Express lines in symmetric form

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{1}$$

$$\frac{x-1}{1} = \frac{y-0}{5} = \frac{z-0}{-1}$$

so direction vectors are $\underline{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

$\underline{u}.\underline{v} = 1+5-1 = 5$ $|\underline{u}| = \sqrt{3}$ $|\underline{v}| = \sqrt{1+25+1} = \sqrt{27}$

$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|} = \frac{5}{\sqrt{3}\sqrt{27}} = \frac{5}{9}$$

$$\theta = \cos^{-1}\left(\frac{5}{9}\right) = 56.3^\circ$$

Maths in Action Book 3 Page 66 Exercise 9A Questions 4 , 5

Page 70 Exercise 11 Questions 1 part (iii)(a), (b)

p298 Ex 15.8 ③ ④ ⑤ part

Cfe p302 Ex 15.11 ① (iii) (a) (b)

Finding the Angle Between Two Planes

The angle **between two planes** is defined to be the **angle between their normal vectors**.

Example
Find the angle between the planes **x + 2y + z = 0** and **x + y = 0**

normal vectors are $\underline{n}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\underline{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\underline{n}_1 . \underline{n}_2 = 1+2$
$= 3$

$|\underline{n}_1| = \sqrt{1+4+1}$
$= \sqrt{6}$

$|\underline{n}_2| = \sqrt{1+1}$
$= \sqrt{2}$

$$\cos\theta = \frac{\underline{n}_1 . \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$$
$$= \frac{3}{\sqrt{6}.\sqrt{2}}$$
$$= \frac{3}{\sqrt{12}}$$
$$= \frac{3}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{2}$$

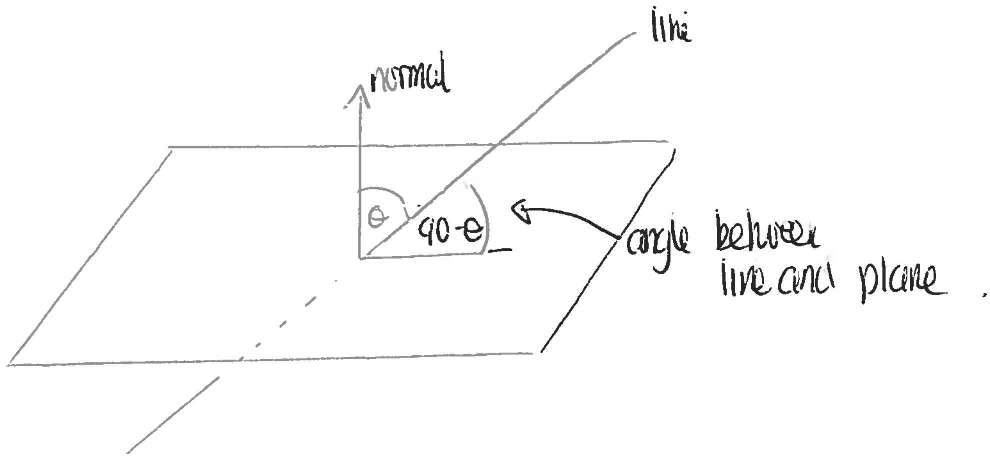
$$\theta = \cos^{-1}\frac{\sqrt{3}}{2}$$
$$= 30^\circ$$

Maths in Action Book 3 Page 59 Exercise 7A Questions 1 , 2 , 3

CfP. p293 EX15.6

Finding the Angle Between A Line and A Plane

The angle between a line and a plane is **(90 - θ)** where **θ is the angle between the line and the normal to the plane.**



Note

(90 - θ) is the smallest angle between the line and the plane

Example

Find the angle between the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ and the plane x + z = 0

direction vector $\underline{d} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ normal vector $\underline{n} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{d}.\underline{n} = 3$ $|\underline{d}| = \sqrt{6}$ $|\underline{n}| = \sqrt{2}$

$\cos\theta = \frac{3}{\sqrt{12}}$

$= \frac{\sqrt{3}}{2}$

$\theta = 30^\circ$

angle between line and plane $= 90 - 30$

$= 60^\circ$

Tips for success.....

angle between two lines -> find angle between direction vectors
angle between two planes -> find angle between normal vectors
angle between line and plane -> $(90-\theta)$
where θ (acute) is angle between direction vector and normal vector

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

Maths in Action Book 3 Page 68 Exercise 10 Questions 1(a), (b) , (c), (d) , 2(a),

Me p300 Ex 15.10 ① part (ii)

The Intersection Of Two Lines

In 3D two lines may be parallel, may intersect or be skew (neither parallel nor intersecting)

Example

$$\frac{x+9}{4} = \frac{y+5}{1} = \frac{z+1}{-2} \qquad (1)$$

$$\frac{x-8}{-5} = \frac{y-2}{-4} = \frac{z-5}{8} \qquad (2)$$

$$\frac{x-8}{-5} = \frac{y-2}{-4} = \frac{z+15}{8} \qquad (3)$$

Consider a) Intersection of (1) and (2)
b) Intersection of (1) and (3)
(Always check for parallel lines e.g. (2) and (3) are parallel – same direction vector)

(a) intersection of $\frac{x+9}{4} = \frac{y+5}{1} = \frac{z+1}{-2}$ $(=t)$ and $\frac{x-8}{-5} = \frac{y-2}{-4} = \frac{z-5}{8}$ $(=s)$

use different letters for parameters

Not parallel

Write in parametric form

Line ① $x = 4t-9$, $y = t-5$, $z = -2t-1$

Line ② $x = -5s+8$, $y = -4s+2$, $z = 8s+5$

For intersection

$$4t-9 = -5s+8 \quad \ldots ①$$
$$t-5 = -4s+2 \quad \ldots ②$$
$$-2t-1 = 8s+5 \quad \ldots ③$$

* Solve ① and ② then check in ③

$$4t+5s = 17 \quad \ldots ①$$
$$t+4s = 7 \quad \ldots ②$$

① $4t+5s = 17$

② ×4 $4t+16s = 28$

Subtract $11s = 11$

$s = 1 \Rightarrow t = 3$

Check in ③ LHS $= -2t-1 = -7$ RHS $= 8s+5 = 13$

Does not work $\Rightarrow$ lines do not intersect

$\Rightarrow$ lines are skew.

(b) $\frac{x+9}{4} = \frac{y+5}{1} = \frac{z+1}{-2}$ $(= t)$ and $\frac{x-8}{-5} = \frac{y-2}{-4} = \frac{z+15}{8}$ $(= \lambda)$

Parametrically

$x = 4t - 9$ $x = -5\lambda + 8$

$y = t - 5$ $y = -4\lambda + 2$

$z = -2t - 1$ $z = 8\lambda - 15$

Point of intersection

$4t - 9 = -5\lambda + 8$ ... ①

$t - 5 = -4\lambda + 2$ ... ②

$-2t - 1 = 8\lambda - 15$ ... ③

Solve ① and ②

$4t + 5\lambda = 17$ ... ①

$t + 4\lambda = 7$ ... ②

① $4t + 5\lambda = 17$

② ×4 $4t + 16\lambda = 28$

Subtract $11\lambda = 11$

$\lambda = 1 \Rightarrow t = 3$

Check in ③ LHS $= -6 - 1 = -7$ RHS $= 8 - 15 = -7$

Works so there is a point of intersection

When $t = 3$ $x = 12 - 9 = 3$ $y = 3 - 5 = -2$ $z = -6 - 1 = 7$

point $(3, -2, -7)$

Maths in Action Book 3 Page 70 Exercise 11 Questions 1(a), 2 , 3

CfE p302 Ex 15.11 Q① parts (i) (ii) ② ③

24

Intersection of Two Planes

Two non-parallel planes will always intersect in a straight line.

Example

Find the equation of the line of intersection of the planes **3x – 5y + z = 8** and **2x –3y +z = 3**

We need ① a direction vector for the line
② a point on the line

① direction vector is perpendicular to both normal vectors

$$\begin{pmatrix}3\\-5\\1\end{pmatrix} \times \begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -5 & 1 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \underline{i}\,(-5+3) - \underline{j}\,(3-2) + \underline{k}\,(-9+10)$$

$$= -2\underline{i} - \underline{j} + \underline{k}$$

② point → can use any point on both planes.

let $z=0$ $\quad 3x-5y=8$
$\quad 2x-3y=3$

could let $x=0$ or $y=0$ instead to find a point on one of the axes

Solve $\quad 6x-10y=16$
$\quad 6x-9y=9$
$\quad y=-7 \Rightarrow x=-9$
$(-9,-7,0)$

line $\quad \dfrac{x+9}{-2} = \dfrac{y+7}{-1} = \dfrac{z}{1}$

Maths in Action Book 3 Page 72 Exercise 12 Questions 1 , 2

CfE p303 Ex 15.12

Intersection of Three Planes

To find the intersection of three planes we solve their equations simultaneously. Since we are solving three equations in three unknowns we use Gaussian elimination.

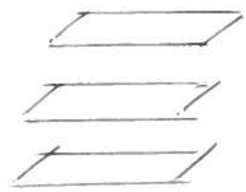
There are three possible outcomes.

1. The system solves uniquely, then there is a point of intersection

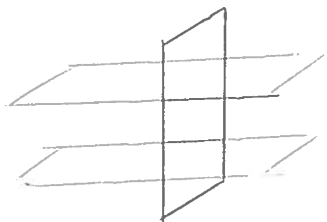
2. One redundant row implies a single line of intersection.

3. An inconsistent row means that there is no solution.

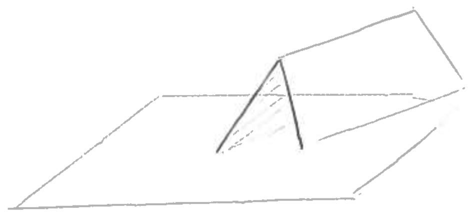
   Either
   - the three planes are parallel



   - two of the planes are parallel



   - One plane is parallel to the line of intersection of the other two.



Example

Consider the intersection of the planes

$$x + 2y - 2z = -7$$
$$x - 2y + z = 6$$
$$3x + 2y - 3z = -8$$

Normal vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ so no planes are parallel.

Use Gaussian elimination to solve.

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 1 & -2 & 1 & 6 \\ 3 & 2 & -3 & -8 \end{array}\right)$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 0 & -4 & 3 & 13 \end{array}\right)$$

$$R_3 - R_2 \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ infinite solutions.
One redundant row $\Rightarrow$ line of intersection.

let $z = t$

(R2) $-4y + 3z = 13$

$y = \dfrac{13 - 3t}{-4}$, $y = \dfrac{3}{4}t - \dfrac{13}{4}$

(R1) $x + 2y - 2z = -7$

$x + \dfrac{3}{2}t - \dfrac{13}{2} - 2t = -7$

$x = \dfrac{1}{2}t - \dfrac{1}{2}$

so
$$\left.\begin{array}{l} x = \frac{1}{2}t - \frac{1}{2} \\ y = \frac{3}{4}t - \frac{13}{4} \\ z = t \end{array}\right\}$$

or in symmetric form

$$\frac{x + \frac{1}{2}}{\frac{1}{2}} = \frac{y + \frac{13}{4}}{\frac{3}{4}} = \frac{z}{1}$$

$$\frac{2x + 1}{1} = \frac{4y + 13}{3} = z$$

Intersection of a Line and a Plane

There are three possible cases

(i) The line is in the plane then intersection is the line itself.

(ii) The line is not in the plane then they intersect at a point.

(iii) The line and plane are parallel and will so they will not intersect.

Check for (i) and (iii) - If the line is in the plane or parallel to the plane then the dot product of the direction vector and the normal will be 0. (i.e. direction vector for line and normal to plane will be perpendicular).

If the point on the line also lies in the plane then we have (i) if it does not then we have (iii).

Example

1) Find the point of intersection of the line $\frac{x-7}{3} = \frac{y-11}{4} = \frac{z-24}{13}$ and the plane $6x + 4y - 5z = 28$

Check $\begin{pmatrix} 3 \\ 4 \\ 13 \end{pmatrix} . \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}$

$= 18 + 16 - 65$

$\neq 0$ so line and plane do intersect.

Write in parametric form

$x = 3t + 7$

$y = 4t + 11$

$z = 13t + 24$

Substitute into equation of plane.

$6x + 4y - 5z = 28$

$6(3t+7) + 4(4t+11) - 5(13t+24) = 28$

$18t + 42 + 16t + 44 - 65t - 120 = 28$

$-31t = 62$

$t = -2$

Point of intersection
When $t = -2$

$x = 3(-2) + 7$
$= 1$

$y = 6(-2) + 11$
$= 3$

$z = 13(-2) + 24$
$= -2$

$(1, 3, -2)$.

Example 2

Consider the intersection of the line $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z}{4}$ and the plane

$2x - 8y + 5z = 14$

Check $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}$
$= 4 - 24 + 20$
$= 0.$

So either line is in plane or parallel to plane.
Check if point ~~is~~ on line ie $(3, -1, 0)$ is also on plane.

Substitute in $2x - 8y + 5z$
$= 6 + 8 + 0$
$= 14$

So point lies on plane.
$\Rightarrow$ line lies on plane.
$\Rightarrow$ intersection is line itself

$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z}{4}$

Tips for success.....

- intersection of lines → write in parametric form
  make 3 equations, solve
- intersection of 2 planes → gives line (if not parallel) → use direction vector $\underline{n}_1 \times \underline{n}_2$
  any point (eg let $z=0$)
- intersection of 3 planes → use Gaussian elimination
- intersection of line and plane → write line in parametric form
  and substitute into plane.

Maths in Action Book 3 Page 68 Exercise 10 Questions 1 part (i)(a), (b) (c), (d) , 4(a) , (c)

CfE p300 Ex 15.10 Q 1 part (i)
2 (a)
4 (a)

<u>Exam Question</u>

(*a*) The line *L* with equation $\dfrac{x+5}{-2} = \dfrac{y-2}{4} = \dfrac{z+1}{3}$ meets the plane π_1 with equation $4x + 2y - z = 3\alpha$ at the point *P*. Find, in terms of α, the coordinates of *P*.

(*b*) The point *P* also lies on the plane π_2 with equation $2x - y - 3z = 42$. Find the value of α and use this value to state the coordinates of *P* and the equation of the plane π_1.

(a) Write line in parametric form

$x = -2t - 5$
$y = 4t + 2$
$z = 3t - 1$

Substitute in $4x + 2y - z = 3\alpha$

$-8t - 20 + 8t + 4 - 3t + 1 = 3\alpha$
$-3t - 15 = 3\alpha$
$3t = -3\alpha - 15$
$t = -\alpha - 5$

so $x = -2(-\alpha-5)-5$
$= 2\alpha+5$

$y = 4(-\alpha-5)+2$
$= -4\alpha-18$

$z = 3(-\alpha-5)-1$
$= -3\alpha-16$

point $(2\alpha+5, -4\alpha-18, -3\alpha-16)$

(b) Substitute point into equation

$2x-y-3z = 42$

$4\alpha+10+4\alpha+18+9\alpha+48 = 42$

$17\alpha = -34$

$\alpha = -2$

$P(1, -10, -10)$

$\pi_1 \quad 4x+2y-z = -6.$