Time 1hr 45minutes

Cfe AH Mathematics

Unit 1 Practice Course Assessment Paper 3

1 (a) Find partial fractions for $\dfrac{1-x}{(1+x)(1+x^2)}$. (4)

(b) Hence, evaluate $\displaystyle\int_0^1 \frac{1-x}{(1+x)(1+x^2)}\,dx.$ (5)

2 Use the substitution $x = 2\sin\theta$ to evaluate the definite integral:-

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{(4-x^2)}}\,dx.$$ (5)

3 (a) A function is defined implicitly by :- $\ln(xy) = x + y$.

Find its derivative, $\dfrac{dy}{dx}$. (3)

(b) A curve has parametric equations:-

$$x = \cos^3\theta, \quad y = \sin^3\theta.$$

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point with parameter θ. (5)

(ii) Prove that the curve has a stationary point when $\theta = \pi$, and investigate its nature. (3)

(c) Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for the function:-

$$y = (\sin x)^{\tan x}, \qquad (\sin x > 0).$$ (4)

4 In a certain electrical circuit, the current i satisfies the differential equation:-

$$L\frac{di}{dt} = E - Ri.$$

(a) Solve this equation to show that $E = Ri + Ae^{-\frac{Rt}{L}}$ (5)

(b) Show that, if the current initially is zero, then at time t, i is defined by:-

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$ (4)

5 Obtain the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 7\sin x + 17\cos x. \qquad 7$$

Hence find the particular solution for which $y = 3$ and $\frac{dy}{dx} = \frac{5}{2}$ when $x = 0$. 3

6 (a) Use integration by parts to find $\int \ln x \, dx$ and $\int (\ln x)^2 \, dx$ (4)

(b) If $I_n = \int (\ln x)^n \, dx$, show that :-

$$I_n = x(\ln x)^n - nI_{n-1} \qquad (2)$$

(c) Use the above results to show that:-

$$\int_1^e (\ln x)^4 \, dx = 9e - 24. \qquad (4)$$

CfE AH Maths Unit 1 Practice Course Assessment Paper 3 Mark Scheme.

1 a	ans : $\frac{1}{1+x} - \frac{x}{1+x^2}$		
	•1	knows how to find partial fractions	•1 $\frac{1-x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$
	•2	knows how to find A, B and C	•2 $1-x = A(1+x^2)+(Bx+C)(1+x)$
	•3	finds A	•3 A = 1
	•4	finds B and C	•4 B = −1, C = 0
	4 marks		
b	ans :	**ln√2 or ½ln2**	
	•1	knows to express integral in P.F's.	•1 $\int_0^1 \frac{1}{1+x} - \frac{x}{1+x^2}\,dx$
	•2	knows how to integrate $\frac{1}{1+x}$	•2 ln(1+x)
	•3	knows how to integrate $\frac{x}{1+x^2}$	•3 $\frac{1}{2}\ln(1+x^2) = \ln\left(\sqrt{(1+x^2)}\right)$
	•4	knows how to evaluate	•4 (ln2 − ln√2) − (ln1 − ln1)
	•5	simplifies	•5 ln√2
	5 marks		

2	ans :	1	
	•1	knows to represent as integral in θ	•1 $x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta$
	•2	knows to represent as integral in θ	•2 $\sqrt{(4-4\sin^2\theta)} = 2\cos\theta$
	•3	knows to change limits of integration	•3 $x = 0 \Rightarrow \theta = 0$ and $x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$
	•4	starts to integrate the new integral	•4 $\int_0^{\frac{\pi}{3}} 2\sin\theta\, d\theta$
	•5	answer	•5 $[-2\cos\theta]_0^{\frac{\pi}{3}} = [-2\cos\frac{\pi}{3}] - [-2] = 1$
	5 marks		

Qu	Marking Scheme Give 1 mark for each •	Illustration of evidence for awarding a mark for each •
3 a	ans : $\frac{dy}{dx}=\frac{xy-y}{x-xy}$	
	•1 differentiates implicitly $\ln(xy)$	•1 $\frac{1}{xy}\left(y+x\frac{dy}{dx}\right)$
	•2 differentiates implicitly $x+y$	•2 $1+\frac{dy}{dx}$
	•3 rearrange	•3 $\frac{dy}{dx}=\frac{xy-y}{x-xy}$
	3 marks	
b(i)	ans : $-\tan\theta$, $\frac{1}{3}\sec^4\theta\,\mathrm{cosec}\,\theta$	
	•1 finds $\frac{dx}{d\theta}$	•1 $-3\cos^2\theta\sin\theta$
	•2 finds $\frac{dy}{d\theta}$	•2 $3\sin^2\theta\cos\theta$
	•3 finds $\frac{dy}{dx}=\frac{dy}{d\theta}/\frac{dx}{d\theta}$	•3 $\frac{dy}{dx}=-\frac{3\sin^2\theta\cos\theta}{3\cos^2\theta\sin\theta}=-\tan\theta$
	•4 knows how to find $\frac{d^2y}{dx^2}$	•4 $\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}=\frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$
	•5 simplifies	•5 $-\sec^2\theta \times \frac{d\theta}{dx}=\frac{1}{3}\sec^4\theta\,\mathrm{cosec}\,\theta$
	5 marks	
b(ii)	ans : **proof, Max S. Pt,**	
	•1 knows to replace θ by π in $\frac{dy}{dx}$	•1 $\frac{dy}{dx}=-\tan\pi=0$ i.e. S. Pt at $\theta=\pi$
	•2 knows to find sign of $\frac{d^2y}{dx^2}$ or table of signs	•2 $\frac{d^2y}{dx^2}$ is undefined.
	•3 confirms maximum with table of signs	•3 $\begin{array}{c\|ccc} \theta & <- & \pi & -> \\ \hline \frac{dy}{dx} & + & 0 & - \end{array}$
	3 marks	
c	ans : $(\sin x)^{\tan x}\left(\sec^2 x\ln(\sin x)+1\right)$	
	•1 knows to take ln of both sides	•1 $\ln y=\ln(\sin x)^{\tan x}$
	•2 simplifies for differentiation	•2 $\ln y=\tan x\ln(\sin x)$
	•3 knows how to differentiate implicitly	•3 $\frac{1}{y}\frac{dy}{dx}=\sec^2 x\ln(\sin x)+\tan x\frac{\cos x}{\sin x}$
	•4 find $\frac{dy}{dx}$	•4 $\frac{dy}{dx}=(\sin x)^{\tan x}\left(\sec^2 x\ln(\sin x)+1\right)$
	4 marks	

4 a	**ans : Proof**	
	•1 knows to separate the variables	•1 $\frac{di}{E-Ri}=\frac{dt}{L}$
	•2 integrates	•2 $-\frac{1}{R}\ln(E-Ri)=\frac{t}{L}+c$
	•3 handle the R	•3 $\ln(E-Ri)=-\frac{Rt}{L}+C$
	•4 takes exponential of both sides	•4 $E-Ri=e^{-\frac{Rt}{L}}\times e^{C}$
	•5 completes proof	•5 $E=Ri+Ae^{-\frac{Rt}{L}}$
	5 marks	
		cont'd ....

4 b	**ans: Proof**	
	•1 knows to replace $t=0, i=0$	•1 $E=Ae^{0} \Rightarrow A=E$
	•2 substitutes for A	•2 $E=Ri+Ee^{\frac{-Rt}{L}}$
	•3 begins to simplify	•3 $Ri=E-Ee^{\frac{-Rt}{L}}$
	•4 completes the proof	•4 $i=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)$
	4 marks	

	Give one mark for each •	Illustrations for awarding each mark
5	**ans:** $y = Ae^{\frac{3}{2}x} + Bxe^{\frac{3}{2}x} - \sin x + \cos x$	
	7 marks	
	• correct auxiliary equation • solves auxiliary equation correctly	• $4m^2 - 12m + 9 = 0$ • $m = \frac{3}{2}(twice)$
	• correct complementary function	• $y = Ae^{\frac{3}{2}x} + Bxe^{\frac{3}{2}x}$
	• correct form of particular integral	• $y = C\sin x + D\cos x$
	• continues correctly • correct particular integral	• $C = -1 or D = 1$ • $y = -\sin x + \cos x$
	• correct general solution	• $y = Ae^{\frac{3}{2}x} + Bxe^{\frac{3}{2}x} - \sin x + \cos x$
	ans: $y = 2e^{\frac{3}{2}x} + \frac{1}{2}xe^{\frac{3}{2}x} - \sin x + \cos x$	
	3 marks	
	• correct value for A • correct derivative	• $A = 2$ • $\frac{dy}{dx} = 3e^{\frac{3}{2}x} + Be^{\frac{3}{2}x} + \frac{3}{2}Bxe^{\frac{3}{2}x} - \cos x - \sin x$
	• correct value for B	• $B = \frac{1}{2}$

6		
a	**ans : $x\ln x - x + c$, $x(\ln x)^2 - 2x\ln x + 2x + c$**	
	•1 knows how to use integration by parts	•1 $x\ln x - \int 1dx$
	•2 completes the integration	•2 $x\ln x - x + c$
	•3 knows how to use integration by parts	•3 $x(\ln x)^2 - 2\int \ln x\, dx$
	•4 completes the integration	•4 $x(\ln x)^2 - 2(x\ln x - x) + c$
	4 marks	
b	**ans : Proof**	
	•1 knows how to use integration by parts	•1 $x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$
	•2 completes the proof	•2 $x(\ln x)^n - nI_{n-1}$
	2 marks	
c	**ans : $9e - 24$**	
	•1 knows to express I_4 in terms of I_3	•1 $x(\ln x)^4 - 4I_3$
	•2 knows to express I_3 in terms of I_2	•2 $x(\ln x)^3 - 3I_2$
	•3 knows to express I_2 in terms of (a) above	•3 $x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x$
	•4 evaluates	•4 $[e(\ln e)^4 - 4e(\ln e)^3 + 12e(\ln e)^2 - 24e\ln e + 24e]$ $- [24] = 9e - 24$
	4 marks	