

Cfe AH Mathematics

Unit 1 Practice Course Assessment Paper 2

1. Find $\frac{dy}{dx}$ and simplify as far as possible :

(a) $y = \tan^{-1}(e^x)$ (3)

(b) $y = \frac{\ln(\sin x)}{(1+x)^2}$ (4)

(c) $x = 3t^2 - 3t^{-4}$, $y = -2t^3 + 4t^{-3}$ (4)

(d) $y = x^x$ (4)

(2) (a) Obtain partial fractions for $\frac{x^2 + 3x - 2}{x^2 + 3x + 2}$. (5)

(b) Hence show that $\int_1^2 \frac{x^2 + 3x - 2}{x^2 + 3x + 2} dx = 1 + 4 \ln\left(\frac{8}{9}\right)$ (4)

(3) Use integration by parts to prove that $\int_0^1 x^2 e^x dx = e - 2$. (4)

(4) Use the substitution $x = 1 - \sin t$ to evaluate $\int_{\frac{\pi}{2}}^1 \frac{dx}{\sqrt{(2x-x^2)}}$. (5)

(5) Solve the first order differential equation $\frac{dy}{dx} = \frac{2y^3 - 1}{6xy^2}$, and find the particular solution, given that $y = 2$ when $x = 5$. (5)

(6) Solve the differential equation

$$(x+1) \frac{dy}{dx} - 3y = (x+1)^4$$

given that $y = 16$ when $x = 1$, expressing the answer in the form $y = f(x)$.

6

(7) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 5y = 4 \cos x.$$

6

Hence determine the solution which satisfies $y(0) = 0$ and $y'(0) = 1$.

4

Unit 1 AH Practice Course Assessment Paper 2.
Mark Scheme

Qu	Marking Scheme Give 1 mark for each •	Illustration of evidence for awarding a mark for each •
1a	ans : $\frac{e^x}{1+e^{2x}}$ <ul style="list-style-type: none"> •¹ knows to use the chain rule •² differentiates $\tan^{-1}(e^x)$ •³ completes simplification 3 marks	<ul style="list-style-type: none"> •¹ $\frac{d}{dx}(e^x) \frac{d}{dx}(\tan^{-1}(e^x))$ •² $\frac{1}{1+e^{2x}}$ •³ $\frac{e^x}{1+e^{2x}}$
1b	ans : $\frac{(1+x)\cot x - 2\ln(\sin x)}{(1+x)^3}$ <ul style="list-style-type: none"> •¹ knows to use the quotient rule •² differentiates $\ln(\sin x)$ •³ differentiates $(1+x)^2$ •⁴ simplifies 4 marks	<ul style="list-style-type: none"> •¹ $\frac{(1+x)^2 \frac{\cos x}{\sin x} - 2(1+x)\ln(\sin x)}{(1+x)^4}$ •² $\frac{\cos x}{\sin x} = \cot x$ •³ $2(1+x)$ •⁴ $\frac{(1+x)\cot x - 2\ln(\sin x)}{(1+x)^3}$
1c	ans : $-t$ <ul style="list-style-type: none"> •¹ know to differentiate x and y w.r.t. t •² know how to find $\frac{dy}{dx}$ •³ differentiates expression •⁴ simplifies 4 marks	<ul style="list-style-type: none"> •¹ $\frac{dx}{dt} = 6t + 12t^{-5}$ and $\frac{dy}{dt} = -6t^2 - 12t^{-4}$ •² $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ •³ $\frac{-6t^2 - 12t^{-4}}{6t + 12t^{-5}}$ •⁴ $\frac{-6t(t + 2t^{-5})}{6(t + 2t^{-5})} = -t$
1d	ans : $\frac{dy}{dx} = x^x(\ln x + 1)$ <ul style="list-style-type: none"> •¹ knows to use logarithmic differentiation •² know to differentiate $\ln y$ implicitly •³ know to differentiate $x \ln x$ by prod. rule •⁴ simplifies 4 marks	<ul style="list-style-type: none"> •¹ $\ln y = \ln x^x = x \ln x$ •² $\frac{1}{y} \frac{dy}{dx}$ •³ $\ln x + 1$ •⁴ $y(\ln x + 1) = x^x(\ln x + 1)$

<p>(2)a</p> <p>ans : $1 + \frac{4}{x+2} - \frac{4}{x+1}$</p> <ul style="list-style-type: none"> •1 knows to divide first •2 knows how to find partial fractions •3 knows how to find A and B •4 finds A and B •5 express in P.F's. <p>5 marks</p> <p>b</p> <p>ans: Proof</p> <ul style="list-style-type: none"> •1 knows how to integrate $\frac{4}{x+2} - \frac{4}{x+1}$ •2 simplifies •3 evaluates •4 completes proof <p>4 marks</p>	<ul style="list-style-type: none"> •1 $1 - \frac{4}{(x+2)(x+1)}$ •2 $\frac{4}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$ •3 $4 = A(x+1) + B(x+2)$ •4 $A = -4, B = 4$ •5 $1 + \frac{4}{x+2} - \frac{4}{x+1}$ <ul style="list-style-type: none"> •1 $4\ln(x+2) - 4\ln(x+1)$ •2 $x + 4\ln\left(\frac{x+2}{x+1}\right)$ •3 $\left(2 + 4\ln\left(\frac{4}{3}\right)\right) - \left(1 + 4\ln\left(\frac{3}{2}\right)\right)$ •4 $1 + 4\ln\left(\frac{8}{9}\right)$
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Qu	Marking Scheme Give 1 mark for each •	Illustration of evidence for awarding a mark for each •
(3)	<p>ans : Proof</p> <ul style="list-style-type: none"> •1 knows how to use integration by parts •2 knows to apply integration by parts for a second time •3 completes integration •4 completes the proof <p>4 marks</p>	<ul style="list-style-type: none"> •1 $[x^2 e^x]_0^1 - \int_0^1 2xe^x dx$ •2 $[x^2 e^x]_0^1 - 2\left[xe^x\right]_0^1 - \int_0^1 e^x dx$ •3 $[x^2 e^x - 2xe^x + 2e^x]_0^1$ •4 $[e] - [2]$

Qu	Marking Scheme Give 1 mark for each •	Illustration of evidence for awarding a mark for each •
(4)	<p>ans : $\frac{\pi}{6}$</p> <ul style="list-style-type: none"> •1 knows to find dx in terms of dt •2 finds $\sqrt{(2x-x^2)}$ in terms of t •3 changes the limits of integration •4 forms a new integral •5 integrates and evaluates <p>5 marks</p>	<ul style="list-style-type: none"> •1 $dx = -\cos t dt$ •2 $\sqrt{(2x-x^2)} = \cos t$ •3 $x = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$ and $x = 1 \Rightarrow t = 0$ •4 $-\int_{\frac{\pi}{6}}^0 1 dt$ •5 $[-t]_{\frac{\pi}{6}}^0 = \frac{\pi}{6}$

(5)

$$\text{ans : } y = \left[\frac{3x+1}{2} \right]^{\frac{1}{3}}$$

- ¹ knows to separate the variables
 - ² know $\int \frac{6y^2}{2y^3-1} dy$ and $\int \frac{dx}{x}$
 - ³ simplifies $\ln(2y^3-1) = \ln x + \ln A$
 - ⁴ knows how to find A , putting $x=5, y=2$
 - ⁵ changes subject to y
- 5 marks**

$$\bullet^1 \quad \int \frac{6y^2}{2y^3-1} dy = \int \frac{dx}{x}$$

$$\bullet^2 \quad \ln(2y^3-1) \text{ and } \ln x$$

$$\bullet^3 \quad 2y^3-1 = Ax$$

$$\bullet^4 \quad A = 3$$

$$\bullet^5 \quad y = \left[\frac{3x+1}{2} \right]^{\frac{1}{3}}$$

(6)

(a)

$$(x+1) \frac{dy}{dx} - 3y = (x+1)^4$$

$$\frac{dy}{dx} - \frac{3}{x+1}y = (x+1)^3$$

1

Integrating factor:

$$\text{since } \int \frac{-3}{x+1} dx = -3 \ln(x+1).$$

1

Hence the integrating factor is $(x+1)^{-3}$.

1

$$\frac{1}{(x+1)^3} \frac{dy}{dx} - \frac{3}{(x+1)^4}y = 1$$

1

$$\frac{d}{dx} ((x+1)^{-3}y) = 1$$

$$\frac{y}{(x+1)^3} = \int 1 dx$$

1

$$= x + c$$

 $y = 16$ when $x = 1$, so $2 = 1 + c \Rightarrow c = 1$. Hence

$$y = (x+1)^4$$

1

(7)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4 \cos x$$

$$\text{A.E. is } m^2 + 2m + 5 = 0$$

$$\Rightarrow m = -1 \pm 2i$$

C. F. is $y = e^{-x}(A \cos 2x + B \sin 2x)$

For P.I. try $f(x) = a \cos x + b \sin x$

$$f'(x) = -a \sin x + b \cos x$$

$$f''(x) = -a \cos x - b \sin x$$

Thus

$$(4a + 2b) \cos x + (4b - 2a) \sin x = 4 \cos x$$

$$\Rightarrow a = 2b \Rightarrow 10b = 4$$

$$\Rightarrow b = \frac{2}{5} \text{ and } a = \frac{4}{5}$$

$$y(x) = e^{-x}(A \cos 2x + B \sin 2x)$$

$$+ \frac{2}{5}(2 \cos x + \sin x)$$

$$y(0) = 0 \Rightarrow A + \frac{4}{5} = 0 \Rightarrow A = -\frac{4}{5}$$

$$y'(x) = e^{-x}(-2A \sin 2x + 2B \cos 2x) -$$

$$e^{-x}(A \cos 2x + B \sin 2x) + \frac{2}{5}(\cos x - 2 \sin x)$$

$$y'(0) = 1 \Rightarrow 2B - A + \frac{2}{5} \Rightarrow B = -\frac{1}{10}$$

$$y = \frac{e^{-x}}{10}(-8 \cos 2x - \sin 2x)$$

$$+\frac{2}{5}(2 \cos x + \sin x)$$

1 for auxiliary equation

1 for roots

1 for form of complementary function

1 for derivatives

1 for substitution

Use of a wrong
PI loses 2 of
these 3 marks.

1 for values

1 for value of A

1 for derivative

1 for value of B

1 for final statement

Total 10