SEQUENCES AND SERIES

- A **sequence** is an ordered list of numbers where each term is obtained according to a fixed rule.

  e.g. 1 , 4 , 9 , 16 , ……….

- A **series** is written as $a_1 + a_2 + a_3 + \ldots\ldots$, the terms of which form a sequence

  e.g 1 + 4 + 9 + 16 + …………..

- The ***nth* term** of a sequence or series is denoted by u_n

- A sequence or series can be defined by a recurrence relation where u_{n+1} is given as a function of lower terms.

  eg $u_{n+1} = 3u_n$ and $u_1 = 2$.
  then $u_2 = 3u_1 = 6$ $u_3 = 3u_2 = 18$ etc.

- A first-order recurrence relation is where $u_{n+1} = f(u_n)$, recall Higher $u_{n+1} = 0.8u_n + 15$

  plain u_n.

Advanced Higher Maths : Unit 2
1.2 Applying Algebraic Skills to Sequences and Series

Examples

1) Find the first 3 terms of each sequence. Is it increasing, decreasing or neither?

(a) $U_n = -2n$
(b) $U_n = 2n + 3$
(c) $U_n = 1 + (-1)^n$

(a) $U_1 = -2\times1 = -2$
$U_2 = -2\times2 = -4$
$U_3 = -2\times3 = -6$ decreasing

(b) $U_1 = 2\times1+3 = 5$
$U_2 = 2\times2+3 = 7$
$U_3 = 2\times3+3 = 9$ increasing

(c) $U_1 = 1+(-1)^1 = 1-1 = 0$
$U_2 = 1+(-1)^2 = 1+1 = 2$
$U_3 = 1+(-1)^3 = 1-1 = 0$ neither.

2) Write the following series in expanded form and evaluate.

(a) $\sum_{r=1}^{6} 3r$

(b) $\sum_{r=1}^{5} (-1)^r \left(\frac{3}{r}\right)$

(a) $\sum_{r=1}^{6} 3r = 3\times1 + 3\times2 + 3\times3 + 3\times4 + 3\times5 + 3\times6$
$= 3 + 6 + 9 + 12 + 15 + 18$
$= 63$

(b) $\sum_{r=1}^{5} (-1)^r\left(\frac{3}{r}\right) = (-1)^1\left(\frac{3}{1}\right) + (-1)^2\left(\frac{3}{2}\right) + (-1)^3\left(\frac{3}{3}\right) + (-1)^4\left(\frac{3}{4}\right) + (-1)^5\left(\frac{3}{5}\right)$
$= -3 + \frac{3}{2} - 3 + \frac{3}{4} - \frac{3}{5}$
$= \frac{-87}{20}$.

Arithmetic Sequences and Series

use to test if terms form an arithmetic sequence.

An **arithmetic sequence** or series is one in which

$$u_{n+1} - u_n = u_n - u_{n-1} \qquad \text{for all values of } n$$

i.e there is a common difference, ***d***, between neighbouring terms in the sequence.

(you add or subtract the same number to get the next term)

This is the defining property of an arithmetic sequence.

e.g. for the sequence 1, 4 , 7, 10, 13,…… the common difference is 3 and the sequence is defined as **the arithmetic sequence with first term 1 and common difference 3.**

For an **Arithmetic Sequence or Series** with first term ***a*** and common difference ***d***, the ***nth*** term is given by

$$u_n = a + (n-1)d$$

* * learn.

u_1	u_2	u_3	etc	u_n
a	$a+d$	$a+2d$		$a+(n-1)d$.

Examples

1) Find an expression for u_n and evaluate u_{11} for the sequence 3, 7, 11, 15,……..

Here $d = 4 \quad a = 3$

$u_n = a + (n-1)d$

$u_n = 3 + (n-1)d$

$u_n = 4n - 1$

So $u_{11} = 4 \times 11 - 1$

$= 43$.

2) In an arithmetic sequence $u_5 = 23$ and $u_{12} = 37$. Find an expression for the *nth* term of this sequence.

$u_5 = 23$
$n = 5$

$u_{12} = 37$
$n = 12.$

$u_n = a + (n-1)d$

$23 = a + 4d \quad \ldots (1)$

$u_n = a + (n-1)d$

$u_{12} = a + 11d$

$37 = a + 11d \quad \ldots (2)$

Solve simultaneously

(2) − (1) $\quad 14 = 7d$
$d = 2.$

In (1) $\quad 23 = a + 8$
$a = 15$

nth term $\quad u_n = a + (n-1)d$
$= 15 + (n-1)2 \quad = 13 + 2n.$

3) Given the arithmetic sequence 5, 11, 17, 23,……find the value of n for which $u_n = 113$

Here $a = 5$, $d = 6$ and $u_n = 113.$

$u_n = a + (n-1)d$

$113 = 5 + (n-1)6$

$6(n-1) = 108$

$n - 1 = 18$

$n = 19$

CfE AH Maths in Action Page 151 Exercise 9.1 Questions 1(a)(d)(g), 2(a)(d)(g), 3, 4, 5, 9, 10, 11

Sum to *n* terms of an Arithmetic Sequence or Series

For the general arithmetic series with first term *a* and common difference *d*, the sum to *n* terms, partial sum, is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

* learn.

Proof of above

$$S_n = a + (a+d) + (a+2d) + \dots + (a+(n-2)d) + (a+(n-1)d)$$

Reverse order.

$$S_n = (a+(n-1)d) + (a+(n-2)d) + \dots + (a+d) + a.$$

Add.

$$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + \dots + (2a+(n-1)d) + (2a+(n-1)d)$$

<- - - n lots - - -> of $(2a+(n-1)d)$

$$2S_n = n(2a+(n-1)d)$$

$$S_n = \frac{n}{2}(2a+(n-1)d)$$

Another form of the formula.

$$S_n = \frac{n}{2}(2a+(n-1)d)$$

$$= \frac{n}{2}(a + a + (n-1)d)$$

last term u_n.

$$S_n = \frac{n}{2}(a+l) \quad \text{where } l = \text{last term}$$

Examples

1) For the arithmetic series 3 + 7 + 11 + 15 + ……, find an expression for S_n, the sum to n terms and hence evaluate S_{20}

Here $a = 3$ $d = 4$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{n}{2}(6 + (n-1) \times 4)$

$S_n = \frac{n}{2}(4n + 2)$

$S_n = 2n^2 + n$.

When $n = 20$

$S_{20} = 2 \times 20^2 + 20$

$= 820$

2) Find the sum of the arithmetic series 17 + 19 + 21 +……+ 231

Here $a = 17$, $d = 2$ $u_n = 231$, $n = ?$

Find n first.

$u_n = a + (n-1)d$

$231 = 17 + (n-1)2$

$231 = 15 + 2n$

$2n = 216$

$n = 108.$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{108} = \frac{108}{2}(34 + (180-1).2)$

$= 13392.$

or Using $S_n = \frac{n}{2}(a + l)$

$= \frac{108}{2}(17 + 231)$

$= 13392.$

3) In an arithmetic progression (general term) $u_{10} = 3$ and $S_6 = 76.5$. Find the smallest value of n such that $S_n < 0$.

$u_{10} = 3 \quad \Rightarrow \quad u_{10} = a + (10-1)d$

$3 = a + 9d \quad \ldots (1)$

$S_6 = 76.5 \quad \Rightarrow \quad S_n = \frac{n}{2}(2a + (n-1)d)$

$76.5 = \frac{6}{2}(2a + 5d)$

$2a + 5d = 25.5 \quad \ldots (2)$

$(1) \times 2 \qquad 2a + 18d = 6$

$(2) \qquad 2a + 5d = 25.5$

Subtract $\qquad 13d = -19.5$

$d = -1.5 \quad \Rightarrow \quad a = 16.5$

$S_n < 0$

$\frac{n}{2}(2a + (n-1)d) < 0$

$\frac{n}{2}\left(33 + (n-1)\left(-\frac{3}{2}\right)\right) < 0$

$(\times 4) \qquad 2n\left(33 - \frac{3}{2}(n-1)\right) < 0$

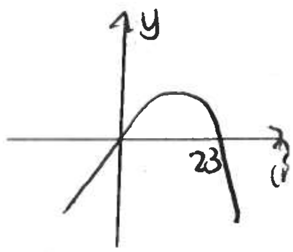
$66n - 3n^2 + 3n < 0$

$69n - 3n^2 < 0$

$23n - n^2 < 0$

$n(23 - n) < 0$

quadratic inequality → graph.



From graph
$n < 0$ or $n > 23$
not possible.

Smallest value of n so that $S_n < 0$ is $\underline{n = 24}$

CfE AH Maths in Action Page 153 Exercise 9.2 Questions 1, 2, 3(a)(c)(e), 4, 6, 8, 9

Geometric Sequences

A geometric sequence or series is one in which there is a **common ratio, *r*,** between neighbouring terms i.e.

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} \quad \text{for all values of } n$$

Use to check if terms form a geometric series.

for example 2 , 6 , 18 , 54,…… is a geometric sequence with **first term 2** and **common ratio 3**

$\times 3$

$a, ar, ar^2, ar^3, \ldots\ldots$

In general for a Geometric Sequence or Series, with first term *a* and common ratio *r*

$$u_n = ar^{n-1}$$

$\star$ $\star$ $\star$

Examples

1. Show that 2sinx, sin2x and sin2xcosx could be the first three terms of a geometric sequence.

$u_1 = 2\sin x$
$u_2 = \sin 2x$
$u_3 = \sin 2x \cos x$.

$$\frac{u_2}{u_1} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$$

$$\frac{u_3}{u_2} = \frac{\sin 2x\cos x}{\sin 2x} = \cos x$$

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \cos x$$

so these terms could form a geometric sequence with $r = \cos x$

2. Find the geometric sequence whose third term is 18 and whose eighth term is 4374.

$u_3 = 18$ $\qquad u_n = ar^{n-1}$

$18 = ar^2 \quad \ldots (1)$

$u_8 = 4374 \quad \Rightarrow \quad 4374 = ar^7 \ldots (2)$

(1) gives $\quad a = \frac{18}{r^2}$.

In (2) $\quad 4374 = \frac{18}{r^2} \cdot r^7$

$r^5 = 243$

$r = 3$

In (1) $\quad a \times 3^2 = 18$

$a = 2$.

sequence $2, 6, 18, 54, \ldots$

3. Given the geometric sequence 3, 6, 12, 24, find n such that $u_n = 196608$

$a = 3 \qquad r = 2 \qquad u_n = 196608$

$u_n = ar^{n-1}$

$196608 = 3 \times 2^{n-1}$

$2^{n-1} = 65536$.

Change to logs.

$n - 1 = \log_2 65536$

$n - 1 = 16$

$n = 17$.

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Sum to *n* terms of a Geometric Sequence or Series

The sum of the terms in a geometric sequence is called a geometric series.

For the general geometric progression with first term a and common ratio r, the sum to n terms, (partial sum or finite sum) is given by

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1}$$

from

$$\begin{aligned}
S_n &= a + ar + ar^2 + \ldots\ldots.. + ar^{n-1} \\
rS_n &= \phantom{a + {}} ar + ar^2 + \ldots\ldots... + ar^{n-1} + ar^n \\
S_n - rS_n &= a - ar^n \\
(1-r)S_n &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

Examples

1) Find the 12th term and S_6 of the geometric sequence 1, 3, 9, 27, …….

$a = 1 \qquad r = 3$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{1(1-3^4)}{1-3}$$

$$= \frac{729-1}{2}$$

$S_6 = 364.$

12th term $\qquad u_n = ar^{n-1}$

$u_{12} = 1 \times 3^{11}$

$= 177147$

2) What is the smallest number of terms that will give a total greater than 6000000 for the sequence 8, 24, 72,…..?

Here $a = 8$

$r = 3$

We want $S_n = \dfrac{a(1-r^n)}{1-r} > 6\,000\,000$

$$\frac{8(1-3^n)}{1-3} > 6\,000\,000$$

$$4(3^n - 1) > 6\,000\,000$$

$$3^n - 1 > 1500000$$

$$3^n > 1500001$$

For $3^n = 1500001$

$$n = \log_3 150001$$

$$n = 12.94$$

so $n = 13$

3) Find two possible geometric series such that $S_2 = 15$ and $S_4 = 255$

$S_2 = 15$ $\qquad S_n = \dfrac{a(1-r^n)}{1-r}$

$$\frac{a(1-r^2)}{1-r} = 15 \qquad (*)(*)$$

$$\frac{a(1-r)(1+r)}{1-r} = 15$$

$$a(1+r) = 15$$

$$a = \frac{15}{1+r} \quad \ldots \; (*)$$

$S_4 = 255$ $\qquad S_n = \dfrac{a(1-r^n)}{1-r}$

$$\frac{a(1-r^4)}{1-r} = 255$$

$$\frac{a(1-r^2)(1+r^2)}{1-r} = 255$$

From $(*)(*)$ $\qquad 15(1+r^2) = 255$

$$1+r^2 = 17$$

$$r^2 = 16$$

$$r = \pm 4$$

In $(*)$

When $r = 4$ $\quad a = \dfrac{15}{5} = 3$ $\qquad$ sequence $3, 12, 48, 192, \ldots$

When $r = -4$ $\quad a = \dfrac{15}{-3} = -5$ $\qquad$ sequence $-5, 20, -80, 320, \ldots$

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Sum To Infinity

If the partial sum S_n tends towards a limit as n tends to infinity, then the limit is called the sum to infinity of the series.

Arithmetic Series

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$na + \frac{n}{2}(nd - d)$$

$$= na + \frac{n^2 d}{2} - \frac{n}{2}d.$$

can be written as

$$= \frac{n^2 d}{2} + na - \frac{n}{2}d$$

$$S_n = n^2\left[\frac{d}{2} + \frac{(a - \frac{d}{2})}{n}\right]$$

$$= \frac{n^2 d}{2} + n\left(a - \frac{1}{2}d\right)$$

$$= n^2\left(\frac{d}{2} + \frac{a - \frac{1}{2}d}{n}\right)$$

and so as $n \to \infty$ $S_n \to n^2 \frac{d}{2}$

and so as $n \to \infty$ $S_n \to \pm\infty$ *depending on value of d*

as $n \to \infty$ this $\to 0$

The sum to infinity for an arithmetic series is undefined. ✱✱

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r} \quad , r \neq 1$$

If $|r| > 1$ *then* $r^n \to \infty$ *as* $n \to \infty$ so S_∞ undefined

If $|r| < 1$ *then* $r^n \to 0$ *as* $n \to \infty$ so $S_n \to \frac{a}{1-r}$

If $r = 1$ *then* $S_n = na \to \pm\infty$ *as* $n \to \infty$ *depending on a* sequence $a, a, a, a \ldots$

If $r = -1$ *then* $S_n = 0$ *for even n and* $S_n = a$ *for odd a, as* $n \to \infty$ sequence $a, -a, a, -a, \ldots$

Hence

The **Sum to Infinity of a Geometric Series** is only defined when $|r| < 1$ and is given by

$$S_\infty = \frac{a}{1-r} \qquad -1 < r < 1$$

*
**

Examples

1) Express $0.0\dot{7}$ as a fraction

$$0.0\dot{7} = 0.07 + 0.007 + 0.0007 + 0.00007 + \dots$$

Geometric series $a = 0.07$
$r = 0.1$

Since $|r| < 1$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{0.07}{1-0.1}$$

$$= \frac{0.07}{0.9}$$

$$= \frac{7}{90}$$

multiply top and bottom by 100 to get rid of decimals.

2) Find the sum to infinity of the geometric series 18 + 12 + 8 + ...

$a = 18$

$r = \frac{12}{18}$

$= \frac{2}{3}$

S_∞ exists since $|r| < 1$

$S_\infty = \frac{a}{1-r}$

$= \frac{18}{1-\frac{2}{3}}$

$= 18 \div \frac{1}{3}$

$= 18 \times 3$

$S_\infty = 54$

3) Given that 18 and 6 are two adjacent terms in an infinite geometric series with sum to infinity of 81, find the first term.

$$U_n = 18$$

$$U_{n+1} = 6$$

$$r = \frac{U_{n+1}}{U_n}$$

$$= \frac{6}{18}$$

$$= \frac{1}{3}$$

$$S_\infty = \frac{a}{1-r}$$

$$81 = \frac{a}{1-\frac{1}{3}}$$

$$81 = a \div \frac{2}{3}$$

$$a = 81 \times \frac{2}{3}$$

$$a = 54$$

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Summary of Formula to Learn for NAB and Exam

Arithmetic Sequence

$$u_{n+1} - u_n = u_n - u_{n-1}$$

$$u_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Geometric Sequence

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}}$$

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$S_\infty = \frac{a}{1-r} \qquad \textit{for } |r| < 1$$

Power Series

We can use polynomials to approximate some functios.

Any series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots$$

is called a **POWER SERIES.**

MacLaurins Theorem

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots}$$

Examples

1.(a) Find a power series for f(x) = cosx

(b) Hence find a power series for cos2x.

(a) $f(x) = \cos x$ $f(0) = 1$

$f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$

$f'''(x) = \sin x$ $f'''(0) = 0$

$f^{iv}(x) = \cos x$ $f^{iv}(0) = 1$

By Maclaurins theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

so $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ (*)

** Note x must be in radians (since using calculus)

(b) In (*) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots$

$= 1 - 2x^2 + \frac{2}{3}x^4 - \ldots$

2(a) Find the first four terms in MacLaurins series for f(x) = e^{2x}
(b) Hence find an expression for e^{2x+3}

$f(x) = e^{2x}$ $\quad f(0) = e^0 = 1$
$f'(x) = 2e^{2x}$ $\quad f'(0) = 2$
$f''(x) = 4e^{2x}$ $\quad f''(0) = 4$
$f'''(x) = 8e^{2x}$ $\quad f'''(0) = 8$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
$$= 1 + \frac{2}{1!}x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 \dots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

3 Find the term in x^4 in the expansion of $e^x(1+2x)$.

expand using Maclaurin

$f(x) = e^x$ $\quad f(0) = 1$
$f'(x) = e^x$ $\quad f'(0) = 1$
$f''(x) = e^x$ $\quad f''(0) = 1$ etc.

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

so $e^x(1+2x) = e^x + 2xe^x$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots\right) + 2x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

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term in x^4

$$\frac{x^4}{24} + \frac{2x^4}{6} = \frac{9}{24}x^4.$$

Basic Expansions

valid for

sinx = $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ $x \in R$

cosx = $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ $x \in R$

e^x = $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ $x \in R$.

ln(1 + x) = $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$ $-1 < x \leq 1$

$\tan^{-1} x$ = $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ $-1 \leq x \leq 1$

We can use these to derive others.

Examples

1. Find a power series for $\ln\frac{(1+x)}{(1-x)}$

We can write

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

Now $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$ $-1 < x \leq 1$ *

To find $\ln(1-x)$ substitute $-x$ for x

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$ $-1 < -x \leq 1$

$-1 \leq x < 1$ *

So $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right)$

$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$ valid for $-1 < x < 1$

* combination of above.

2. Expand $e^{-2x}\sin 3x$ in ascending powers of x as far as x^4.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad x \in R.$$

Replace x by $-2x$

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \dots$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots \qquad x \in R.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad x \in R.$$

Replace x by $3x$

$$\sin 3x = 3x - \frac{9}{2}x^3 + \dots$$

Hence $e^{-2x}\sin 3x = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 \dots\right)\left(3x - \frac{9}{2}x^3 + \dots\right)$

$$= 3x - 6x^2 + 6x^3 - 4x^4 - \frac{9}{2}x^3 + 9x^4 - \dots$$

$$= 3x - 6x^2 + \frac{3}{2}x^3 + 5x^4 + \dots \qquad \text{valid for } x \in R.$$

3. Expand $e^{\sin x}$ as far as x^4.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Substitute $\sin x$ for x

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \frac{\sin^4 x}{4!} + \dots$$

But $\sin x = x - \frac{x^3}{3!} + \dots$

So $e^{\sin x} = 1 + \left(x - \frac{x^3}{3!}\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!}\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3}\right)^3 + \frac{1}{4!}\left(x - \frac{x^3}{3}\right) + \dots$

$$= 1 + x - \frac{x^3}{6} + \frac{x^2}{2} - \frac{x^4}{6} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$$

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