PROOF

Symbol	Meaning
$\Rightarrow$	"implies" ie if ... then eg $x=3 \Rightarrow x^2=9$ (not true backwards)
$\Leftarrow$	"implied by" eg $\lvert x+2\rvert=3$ $\Leftarrow x=1$ (x could be -5 too)
$\Leftrightarrow$	"if and only if" equivalent to eg $x+2=3$ $x=1$

We will consider four methods of proof.

1. Direct Proof
   - start with known truth and by a succession of correct deductions finish with the required result.

2. Proof by Contradiction
   - assume the opposite is true and show that this cannot be true.
   -

3. Proof by Contrapositive

4 Proof by Induction
   Already done

Direct Proof

Example

Prove that if a and b are any two real numbers then a < b implies that $a < \dfrac{a+b}{2}$

We know $a < b$

$$\Leftrightarrow \frac{a}{2} < \frac{b}{2}$$

$$\Leftrightarrow \frac{a}{2} + \frac{a}{2} < \frac{b}{2} + \frac{a}{2}$$

$$a < \frac{a+b}{2}.$$

Example

Prove that if n is odd then $(n+2)^2 - n^2$ is a multiple of 8. (use a direct proof)

n is odd so let $n = (2k+1)$

so $(n+2)^2 - n^2$

$= (2k+1+2)^2 - (2k+1)^2$

$= (2k+3)^2 - (4k^2+4k+1)$

$= 4k^2+12k+9-4k^2-4k-1$

$= 8k+8$

$= 8(k+1)$

Example

Prove that

$$|x+y| \leq |x| + |y|$$

given that

(a) $|x| = +\sqrt{x^2}$

(b) $|xy| = |x||y|$

(c) $xy \leq |xy|$

If (b) and (c) are true.

$\Rightarrow xy \leq |x||y|$

$\Rightarrow 2xy \leq 2|x||y|$

Add x^2+y^2 to both sides.

$x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2$

$(x+y)^2 \leq |x|^2 + 2|x||y| + |y|^2$

$(x+y)^2 \leq (|x|+|y|)^2$

$\sqrt{(x+y)^2} \leq \sqrt{(|x|+|y|)^2}$

$|x+y| \leq |x| + |y|$ by (a)

as required.

Example

Prove that if S(n) is the sum of the first n natural numbers then $S(n) = \frac{1}{2}n(n+1)$

$S(n) = 1 + 2 + 3 + 4 + \dots + (n-1) + n$

Write backwards

$S(n) = n + (n-1) + (n-2) + (n-3) + \dots + 2 + 1$

Add

$2S(n) = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1)$

n lots of $(n+1)$

$2S(n) = n(n+1)$

$S(n) = \frac{1}{2}n(n+1)$ as required.

Example

Prove that $n^3 - n$ is always divisible by 6.

$n^3 - n$

$= n(n^2 - 1)$

$= n(n-1)(n+1)$

$= (n-1)n(n+1)$ Three consecutive numbers

$\Rightarrow$ one must be divided by 2 and one must divide by 3

$\therefore$ $n^3 - n$ must be divisible by $2 \times 3 = 6$

Note we can write $(n^3 - n) \mid 6$ which means $n^3 - n$ is divisible by 6.

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worksheet

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Proof by Contradiction

To prove a conjecture (statement) by contradiction we assume that the conjecture is false and then show that this leads to a conclusion that is false. This then shows that the original conjecture must actually have been true.

Hints

- If assuming an expression is rational write it as $\frac{m}{n}$ where $m, n \in Z$ and they have no common factors.
- If assuming an expression is even write it as 2k where $k \in Z$ must state this too.
- If assuming an expression is odd write it as 2k+1 where $k \in Z$

Examples

1. Prove by contradiction that if m^2 is even then so is m $(m \in N)$

given this so must be true

assume this is false. ie m is odd.

Assume m is odd

ie $m = 2k+1 \quad (k \in Z)$

$m^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 4(k^2 + k) + 1$ which is odd.

This contradicts what we are given (ie m^2 is even)

so m must be even if m^2 is even.

2. Prove by contradiction that $\sqrt{3}$ is irrational.

assume false ie $\sqrt{3}$ is rational.

Assume $\sqrt{3}$ is rational.

ie $\sqrt{3} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and m and n have no common factor

remember method

Square $3 = \frac{m^2}{n^2}$

$m^2 = 3n^2$ $\Rightarrow$ m^2 is a multiple of 3

$\Rightarrow$ m is a multiple of 3

let $m = 3p$

$(3p)^2 = 3n^2$

$9p^2 = 3n^2$

$n^2 = 3p^2$

so n^2 is a multiple of 3

so n is a multiple of 3

so m and n have a common factor 3

This contradicts initial assumption

so $\sqrt{3}$ is irrational

3. Prove by contradiction that $\log_{10} 5$ is irrational.

Assume $\log_{10} 5$ is rational.

ie $\log_{10} 5 = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ where m and n have no common factor.

$\Rightarrow 5 = 10^{\frac{m}{n}}$

$5^n = 10^m$

This is not possible because 5^n has last digit 5
10^n has last digit 0.

This gives a contradiction so the original assumption is false.

ie $\log_{10} 5$ is irrational.

4. Prove by contradiction that $\frac{a+b}{2} \geq \sqrt{ab} \quad \forall \quad a,b \in N$

Assume that $\frac{a+b}{2} < \sqrt{ab}$ for some $a, b \in N$

$\Rightarrow a+b < 2\sqrt{ab}$

$(a+b)^2 < 4ab$ (valid since $a, b \in N$ ie $a, b > 0$)

$a^2 + 2ab + b^2 < 4ab$

$a^2 - 2ab + b^2 < 0$

$(a-b)^2 < 0$

not possible.

$\Rightarrow$ assumption is false.

so $\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a, b \in N$.

Tips for success.....

To prove by contradiction
- Assume and state 'opposite' of what you are trying to prove.
- Show that this does not work.
  - $\Rightarrow$ assumption is wrong.
  - $\Rightarrow$ original statement is true.
- to prove irrational, assume rational in form $\frac{m}{n}$ where m, n have no common factors.

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Counter Examples

To disprove a statement only one counter-example needs to be given (ie one example which does not work)

Examples

1. Find a counter example to disprove the conjecture that if $0 < a < b$ then $a^n < b^n$ for $n < 0$.

$a = \frac{1}{2}$ $b = 1$ $n = -2$

$a^n = \left(\frac{1}{2}\right)^{-2}$
$= 2^2$
$= 4$

$b^n = 1^{-2}$
$= 1$

so $a^n > b^n$
so conjecture is false

2. Find a counter example to disprove the conjecture that $n^2 + n + 41$ is a prime number for all $n \in N$.

Consider $n = 41$

$41^2 + 41 + 41$
$= 41(41 + 1 + 1)$
$= 41 \times 43$ which is not prime
so conjecture is false.
ie $n^2 + n + 41$ is not prime $\forall n \in N$.

Worksheet

The Infinity of Primes

There is an infinite number of primes. * proof by contradiction

Proof

Suppose that there is a finite number of primes.

$p_1, p_2, p_3, \ldots p_n$ for some n.

Consider the number

$$N = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$$

Now N is not divisible by $p_1, p_2, p_3, \ldots p_n$ since on dividing by any one of them leaves a remainder 1. But N must have at least one prime factor so there is a prime number other than $p_1, p_2, \ldots p_n$. This contradicts our initial assumption which must then be false. Hence there is an infinite number of primes.

The Converse

To form the **converse** of an *If-then* sentence, exchange the hypothesis and conclusion.

The converse of

If p, then q,

where p and q are sentences, is

If q, then p.

Clearly, if an *If-then* sentence is true, its converse is not necessarily true.

Example

State the converse of each statement, and then decide whether the converse is true. (Note that each statement is true.)

a) *If a number ends in 5, then it is a multiple of 5.*

If a number is a multiple of 5, then it ends in 5.
False eg 20 is a multiple of 5.

b) *If a number is a multiple of 10, then it ends in 0.*

If a number ends in 0, then it is a multiple of 10.
True.

Example

State the converse of *All right angles are equal.*

All equal angles are right-angles.
False.

"if and only if"

When a statement *If a, then b* and its converse *If b, then a* are both true, we say "*a if and only if b.*"
In other words, *a* is both necessary and sufficient for *b*.

For example,

A triangle is isosceles if and only if the base angles are equal.

This means

If a triangle is isosceles, then the base angles are equal

and conversely,

If the base angles are equal, then the triangle is isosceles.

Example

What does it mean to say, "A number is a multiple of 10 if and only if it ends in 0"?

If a number is a multiple of 10 then it ends in 0 and conversely if a number ends in 0 then it is a multiple of 10.

Example

Which of these *if and only if* statements is true. Explain.

a) A number is divisible by 6 if and only if it is divisible by both 2 and 3.

b) A number is a multiple of 9 if and only if it is a multiple of 3.

c) A fraction is in lowest terms if and only if the numerator and denominator have no common divisors except 1.

(a) True because if a number is divisible by 6. then it is divisible by both 2 and 3 and conversely

(b) False. It is true that if a number is a multiple of 9 then it is a multiple of 3, but the converse is false eg 6.

(c) True - because if a fraction is in lowest terms then the numerator and denominator have no common divisors ex

The Negation of a Statement

The negation of a statement p is a statement saying the opposite to p.
e.g. if p is $x > 3$ then its negation is x is not greater than 3 or $x \le 3$.
We denote the negation of p by ~p.

Example

For each of these statements p state its negation ~p.

(a) p : $x > 0, x \in Z$ — $x \leq 0, x \in Z$

(b) p : $\sqrt{x}$ is rational — $\sqrt{x}$ is not rational ie $\sqrt{x}$ is irrational.

(c) p : x is a negative real — x is not a negative real number. (can't say positive because this would not include 0)

Contra positive

If a sentence has the form:

if *a*, then *b*.

then

if *not-b*, then *not-a*. or

is called it the contrapositive of "if *a*, then *b*." The hypothesis and conclusion are exchanged and contradicted.

We say that a statement and its contrapositive are "logically equivalent." That is a technical way of saying that they mean the same. They will either both be true or both be false.

"if p then q" is true $\Leftrightarrow$ "if ~q then ~p" is true.

Examples

1. State the contrapositive.
   a) If a number ends in 6, then it's even.
   b) If two lines are parallel, then they do not meet.

(a) If a number is not even then it does not end in a 6
(b) If two lines meet then the two lines are not parallel.

2. Prove, by using the contrapositive, that if m, n are integers and $mn = 100$ then either $m \leq 10$ or $n \leq 10$.

Statement If $mn = 100$, $m, n \in \mathbb{Z}$ then either $m \leq 10$ or $n \leq 10$

Contrapositive If $m > 10$ and $n > 10$ then $mn \neq 100$

Proof $m > 10$ and $n > 10$

so $mn > 10 \times 10$

$mn > 100$

so $mn \neq 100$ Contrapositive is true.
Hence statement is true.

3. Prove, by using the contrapositive, that if 7 is not a factor of n^2 then 7 is not a factor of n.

Statement If 7 is not a factor of n^2 then 7 is not a factor of n.

Contrapositive If 7 is a factor of n, then 7 is a factor of n^2.

Proof 7 is a factor of n

so $n = 7m$ for some $m \in N$

$n^2 = 49m^2$

$= 7(7m^2)$

so 7 is a factor of n^2

Contrapositive is true.
so statement is true.

4. Prove, by using the contrapositive, that if x and y are integers and xy is odd then both x and y are odd.

Statement if xy is odd then both x and y are odd.

Contrapositive if x and/or y are even then xy is even.

Proof x and y both even.

$x = 2n \quad y = 2m \qquad n, m \in \mathbb{Z}$

$xy = 4nm$ which is even.

x even, y odd

$x = 2n \quad y = 2m+1$

$xy = 2n(2m+1)$ which is even.

x odd, y even

$x = 2n+1 \quad y = 2m$

$xy = (2n+1)2m$ which is even.

Contrapositive true.

so statement is true.

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