NUMBER THEORY

The Division Algorithm / Euclidean Algorithm

The GREATEST COMMON DIVISOR (gcd) of a and b can be denoted by (a, b) when the context is unambiguous.

Consider

$$203 \div 4 = 50r3$$

so

$$\frac{203}{4} = 50 + \frac{3}{4}$$

203 is the DIVIDEND
4 is the DIVISOR
50 is the QUOTIENT
3 is the REMAINDER

The relationship between these four numbers can also be expressed as
$203 = 4 \times 50 + 3$

In fact, given any positive integers a and b $(b \neq 0)$ there exist unique integers q and r

where $0 \leq r < b$, such that

$$a = bq + r$$

This is the **DIVISION ALGORITHM.**

If $a = bq + r$ then $(a, b) = (b, r)$ ✱

The repeated application of this until the gcd (greatest common divisor) is identified is known as the **EUCLIDEAN ALGORITHM.**

eg greatest common divisor of 36 and 12 is 2.

$36 = 2 \times 12 + 10$ $\quad (34, 12) = (12, 10)$
$12 = 1 \times 10 + 2$ $\quad (12, 10) = (10, 2)$
$10 = 5 \times 2 + 0.$ $\quad (10, 2) = (2, 0)$

so gcd = 2.

NB $(b, 0) = b$
greatest common divisor of b and 0 is b.

Advanced Higher Maths Unit 3
1.4 Applying Algebraic Skills to Number Theory

Examples

1. Find the gcd of 203 and 4.

$203 = 50 \times 4 + 3$
$4 = 1 \times 3 + 1$
$3 = 3 \times 1 + 0$

$(203, 4) = (4, 3)$
$= (3, 1)$
$= (1, 0)$ keep going til get 0
$= 1$

The greatest common divisor of 203 and 4 is one.

2. Find the gcd of 132 and424

$423 = 3 \times 132 + 28$
$132 = 4 \times 28 + 20$
$28 = 1 \times 20 + 8$
$20 = 2 \times 8 + 4$
$8 = 2 \times 4 + 0.$

$(424, 132) = (132, 28)$
$= (28, 20)$
$= (20, 8)$
$= (8, 4)$
$= (4, 0)$

gcd $= 4$

If the gcd of two numbers is 1, they are said to be **relatively prime or co-prime.**

Example

Find the gcd of 140 and 252.

$252 = 1 \times 140 + 112$

$140 = 1 \times 112 + 28$

$112 = 4 \times 28 + 0$

$(252, 140) = (140, 112)$

$= (112, 28)$

$= (28, 0)$

gcd of 140 and 252 is 28

Maths in Action Book 3 p145 Ex 4 Q 1 – 6

The gcd as a Linear Combination

If d is the greatest common divisor of a and b then there exist integers s and t such that d = as + bt

Example

1. Express the gcd of 2695 and 1260 in the form 2695s + 1260t where $s, t \in Z$

Find the gcd first.

$2695 = 2 \times 1260 + 175 \quad \ldots ②$

$1260 = 7 \times 175 + 35 \quad \ldots ①$

$175 = 5 \times 35 + 0$

$(2695, 1260) = (1260, 175)$
$= (175, 35)$
$= (35, 0)$

gcd = 35

Working backwards.

① gives $35 = 1260 - 7 \times 175$

$= 1260 - 7 \times (2695 - 2 \times 1260)$

$= 1260 - 7 \times 2695 + 14 \times 1260$

$= 15 \times 1260 - 7 \times 2695$

$= 2695 \times (-7) + 1260 \times (15)$

so $s = -7$ and $t = 15$

2. Exam Question
(a) Use the Euclidean Algorithm to obtain the greatest common divisor of 3255 and 4785

(b) Hence find the two integers x and y such that 3255x + 4785y = 360

(a)
$4785 = 1 \times 3255 + 1530$
$3255 = 2 \times 1530 + 195$
$1530 = 7 \times 195 + 165$
$195 = 1 \times 165 + \underline{\underline{30}}$ --- (*)
$165 = 5 \times 30 + 15$
$30 = 2 \times 15 + 0$

$(4785, 3255) = (3255, 1530)$
$= (1530, 195)$
$= (195, 165)$
$= (165, 30)$
$= (30, 15)$
$= (15, 0)$

gcd = 15

(b) Write for 30 first from (*) then x12 to get 360.

$30 = 195 - 1 \times 165$
$= 195 - 1 \times (1530 - 7 \times 195)$
$= 8 \times 195 - 1 \times 1530$
$= 8 \times (3255 - 2 \times 1530) - 1 \times 1530$
$= 8 \times 3255 - 17 \times 1530.$
$= 8 \times 3255 - 17 \times (4785 - 1 \times 3255)$
$30 = 25 \times 3255 - 17 \times 4785$

Multiply by 12
$360 = 3255 \times 300 - 4785 \times 204$

- Maths in Action book3 p147 Exercise 5 Q 1 – 7

Number Bases

We are used to working with numbers in base 10. This means that are column headings are

$$10^4 \qquad 10^3 \qquad 10^2 \qquad 10^1 \qquad 10^0$$

So the number 62314 is

10^4	10^3	10^2	10^1	10^0
6	2	3	1	4

i.e. $\quad 62314_{10} = 6\times10^4 + 2\times10^3 + 3\times10^2 + 1\times10^1 + 4\times10^0$

We can write numbers in other bases in a similar way.

Example
For base 5 the column headings are

$$5^4 \qquad 5^3 \qquad 5^2 \qquad 5^1 \qquad 5^0$$

so

$$33224_5 = 3\times5^4 + 3\times5^3 + 2\times5^2 + 2\times5^1 + 4\times5^0$$
$$= 2314_{10}$$

Note
In base 5 the highest digit we can have is 4
In base 2 the highest digit we can have is 1. etc

Example
1. Write the following numbers in base 10.
(a) 101101_2 (b) 1223_4

(a)

2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	1	0	1

$101101_2 = 1\times2^0 + 1\times2^1 + 1\times2^3 + 1\times2^5$
$= 1 + 4 + 8 + 32$
$= 45_{10}.$

(b)

4^3	4^2	4^1	4^0
1	2	2	3

$1223_4 = 1\times4^3 + 2\times4^2 + 2\times4 + 3$
$= 64 + 32 + 8 + 3$
$= 107_{10}$

2. Express 1467_{10} in base 6.

$1467 \div 6 = 244 \text{ r } 3$

$244 \div 6 = 40 \text{ r } 4$

$40 \div 6 = 6 \text{ r } 4$

$6 \div 6 = 1 \text{ r } 0$

$1 \div 6 = 0 \text{ r } 1$

stop at 0

required number * read upwards.

$1467_{10} = 10443_6$.

3. Express 1213_4 in base 5.

Write in base 10 first

$$\begin{aligned}1213_4 &= 1\times4^3 + 2\times4^2 + 1\times4^1 + 3\times4^0\\ &= 64 + 32 + 4 + 3\\ &= 103\end{aligned}$$

$103 \div 5 = 20 \text{ r } 3$

$20 \div 5 = 4 \text{ r } 0$

$4 \div 5 = 0 \text{ r } 4$

$1213_4 = 103_{10} = 403_5$

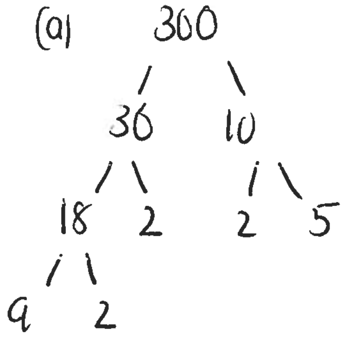
Maths in Action Book 3 p151 Ex7

The Fundamental Theorem of Arithmetic

The fundamental theorem of arithmetic states that every positive integer can be expressed uniquely as the product of one or more primes.

Examples
(a) 360 (b) 37 (c) 416

(a)

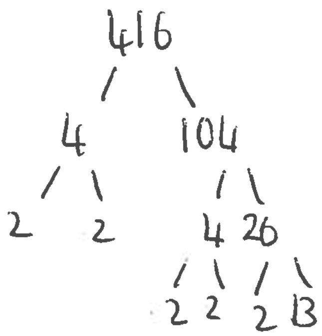


$360 = 2^3 \times 5 \times 9$

(b) 37 prime.

$37 = 37$

(c)



$416 = 2^5 \times 13.$

NAB Style Questions

1. Use the Euclidean algorithm to obtain (1365, 299)
2. Use the Euclidean algorithm to obtain (5187, 760)

Exam Style Question

- Use the Euclidean algorithm to find integers x and y such that $9454x + 6873y = 29$

(1)

$1365 = 4 \times 299 + 169$
$299 = 1 \times 169 + 130$
$169 = 1 \times 130 + 39$
$130 = 3 \times 39 + 13$
$39 = 3 \times 13 + 0$

$(1365, 299) = (299, 169)$
$= (169, 130)$
$= (130, 39)$
$= (39, 13)$
$= (13, 0)$

gcd = 13.

(2)

$5187 = 6 \times 760 + 627$
$760 = 1 \times 627 + 133$
$627 = 4 \times 133 + 95$
$133 = 1 \times 95 + 38$
$95 = 2 \times 38 + 19$
$38 = 2 \times 19 + 0$

$(5187, 760) = (760, 627)$
$= (627, 133)$
$= (133, 95)$
$= (95, 38)$
$= (38, 19)$
$= (19, 0)$

$(5187, 760) = 19$

Find gcd of 9454 and 6873

$$9454 = 1\times 6873 + 2581$$
$$6873 = 2\times 2581 + 1711$$
$$2581 = 1\times 1711 + 870$$
$$1711 = 1\times 870 + 841$$
$$870 = 1\times 841 + 29 \quad \text{-- } (*)$$
$$841 = 29\times 29 + 0$$

$$(9454, 6873) = (6873, 2581)$$
$$= (2581, 1711)$$
$$= (1711, 870)$$
$$= (870, 841)$$
$$= (841, 29)$$
$$= (29, 0)$$

gcd $= 29$.

From $(*)$

$$29 = 870 - 1\times 841$$
$$= 870 - 1\times(1711 - 1\times 870)$$
$$= 2\times 870 - 1\times 1711$$
$$= 2\times(2581 - 1\times 1711) - 1\times 1711$$
$$= 2\times 2581 - 3\times 1711$$
$$= 2\times 2581 - 3\times(6873 - 2\times 2581)$$
$$= 8\times 2581 - 3\times 6873.$$
$$= 8\times(9454 - 1\times 6873) - 3\times 6873$$
$$= 8\times 9454 - 11\times 6873.$$

$x = 8$ and $y = -11$