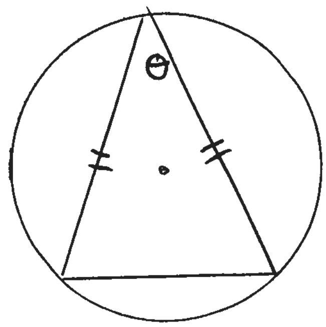
MORE CALCULUS APPLICATIONS

Optimisation

Differentiation can be used to analyse problems and help to find optimum values of systems as long as they can be modelled satisfactorily.

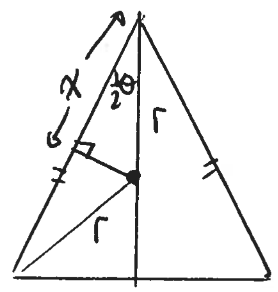
Example

The diagram shows an isosceles triangle inscribed in a circle radius r.



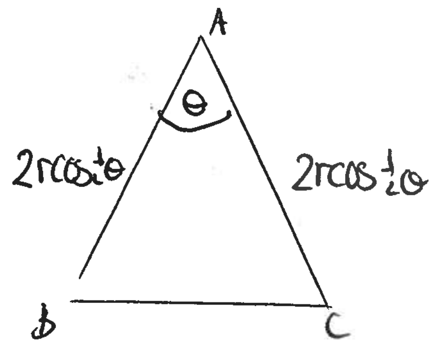
(a) Show that the area of the triangle is given by
$A = r^2 \sin\theta(1+\cos\theta)$
where θ is the angle between the equal sides.

(b) Find the maximum possible area of the triangle.



$\cos\frac{1}{2}\theta = \frac{x}{r}$

$x = r\cos\frac{1}{2}\theta$.



$$\text{Area} = \frac{1}{2}bc\sin A$$
$$= \frac{1}{2} \times 2r\cos\frac{1}{2}\theta \times 2r\cos\frac{1}{2}\theta \cdot \sin\theta$$
$$= 2r^2\cos^2\frac{1}{2}\theta\sin\theta.$$
$$= r^2\sin\theta\left(2\cos^2\frac{1}{2}\theta\right)$$
$$= r^2\sin\theta\left(1+\cos\theta\right)$$

as required.

since $\cos 2\theta = 2\cos^2\theta - 1$
$2\cos^2\theta = \cos 2\theta + 1$
$2\cos^2\frac{\theta}{2} = \cos\theta + 1$

(b) $A = r^2 \sin\theta\,(1+\cos\theta)$ $\quad$ r constant.

Differentiate wrt θ.

$$\frac{dA}{d\theta} = r^2\sin\theta(-\sin\theta) + (1+\cos\theta).r^2\cos\theta$$
$$= -r^2\sin^2\theta + r^2\cos\theta + r^2\cos^2\theta$$
$$= r^2(\cos^2\theta - \sin^2\theta) + r^2\cos\theta.$$
$$= r^2\cos 2\theta + r^2\cos\theta$$
$$= r^2(\cos 2\theta + \cos\theta)$$

For maximum area $\frac{dA}{d\theta} = 0$

$$r^2(\cos 2\theta + \cos\theta) = 0$$
$$\cos 2\theta + \cos\theta = 0$$
$$2\cos^2\theta - 1 + \cos\theta = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$\cos\theta = \frac{1}{2}$ $\qquad$ ~~cos~~ $\cos\theta = -1$

$\theta = \frac{\pi}{3}$ $\qquad$ $\theta = \pi$ not possible in Δ.

Nature

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\frac{dA}{d\theta} = r^2(\cos 2\theta + \cos\theta)$	r^2	0	$-r^2$
	/	—	\

so $\theta = \frac{\pi}{3}$ gives maximum area.

When $\theta = \frac{\pi}{3}$

$A = r^2 \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right)$

$= r^2 . \frac{\sqrt{3}}{2} . \frac{3}{2}$

$= \frac{3\sqrt{3}}{4} r^2$

Worksheet Ex 9
CfE Maths in Action p200 Exercise 11.4 Questions 1, 2, 4, 9

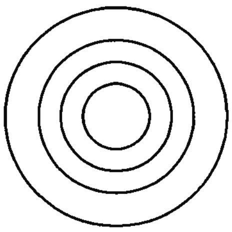
or $\frac{d^2A}{d\theta^2} = -2r^2\sin 2\theta - r^2\sin\theta.$

When $\theta = \frac{\pi}{3}$ $\frac{d^2A}{d\theta^2} = -2r^2\sin\frac{2\pi}{3} - r^2\sin\frac{\pi}{3} = -2r^2.\frac{\sqrt{3}}{2} - r^2\frac{\sqrt{3}}{2}$

$= -3r^2\frac{\sqrt{3}}{2} < 0$ so max

Related Rate Problems – Application of the Chain Rule

When a pebble is dropped into a pool of water a series of concentric circles is produced. Both the radius and area of each circle vary with time, and naturally the area of each circle varies with the radius. The *rate of change of radius* with respect to



time, $\frac{dr}{dt}$, *rate of change of area* with respect to time, $\frac{dA}{dt}$, *rate of change of area* with respect to radius, $\frac{dA}{dr}$, are all related and relationships may made using the chain rule:

e.g. $$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

Many such relationships can be found and are of value when solving problems involving rates.

Examples

1. A stone is dropped into a pool of water, creating circular ripples.
Find the rate at which the area of the outer circle is increasing (in cm/s), at the point where the radius is 4 cm.
The rate at which the radius is changing is $\frac{3}{2}$ cm/s.

Circle $A = \pi r^2$ (*)

We want $\frac{dA}{dt}$

Differentiate (*) w.r.t time

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We know $\frac{dr}{dt} = \frac{3}{2}$. and $r = 4$

so $$\frac{dA}{dt} = 2\pi \times 4 \times \frac{3}{2}$$

$$\frac{dA}{dt} = 12\pi \quad \text{cm/s.}$$

2. A spherical balloon is blown up such that its volume increases at the constant rate of 8 cm^3/s.
At what rate is the radius increasing at the instant the radius is 4 cm ?

Volume of sphere $V = \frac{4}{3}\pi r^3$

Differentiate wrt time

$$\frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 . \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

We know $\frac{dV}{dt} = 8$ $r = 4$

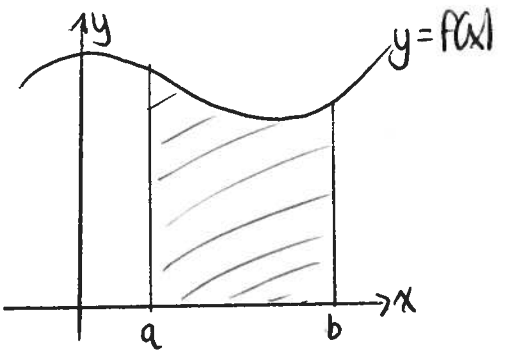
$$8 = 4 \times \pi \times 4^2 \times \frac{dr}{dt}$$

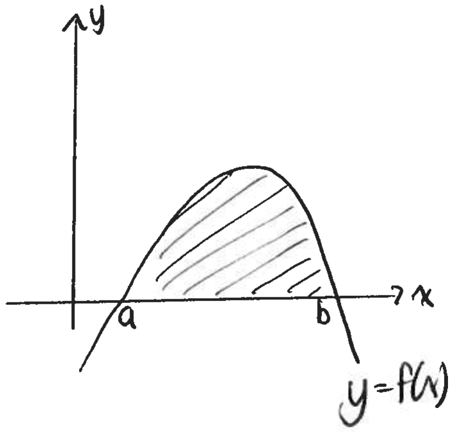
$$\frac{dr}{dt} = \frac{8}{\pi \times 4^3}$$

$$\frac{dr}{dt} = \frac{1}{8\pi}$$

CfE Maths in Action p196 Exercise 11.3 Questions 1 , 2,3(a), 5 to 10

Areas Between the Curve and the x-axis





$$\text{Area} = \int_a^b f(x)dx$$

Note Always sketch graph first.
Areas below x-axis give negative answer.
Work out areas above and below x-axis separately.

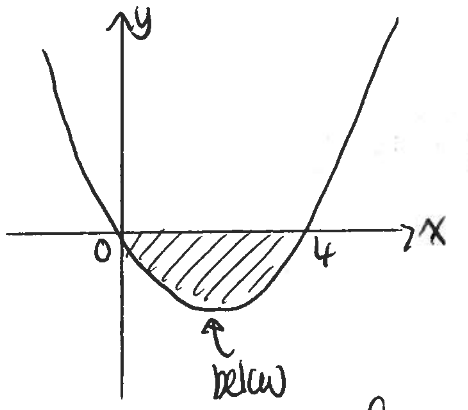
Examples

1. Calculate the area bounded by $y = x^2 - 4x$ and the x-axis.

Sketch parabola U

cuts x-axis $x(x-4) = 0$

$x = 0 \quad x = 4$



so area $= -\int \dots dx$

$$\text{area} = -\int_0^4 (x^2 - 4x)dx$$

$$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= -\left\{\left(\frac{64}{3} - \frac{64}{2}\right) - 0\right\}$$

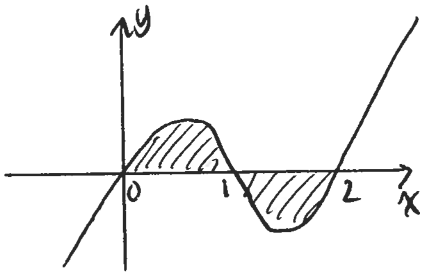
$$= 10\tfrac{2}{3} \text{ square units}$$

2. Find the area between the curve y = x(x – 1)(x – 2) and the x-axis.

$+x^3$ 

cuts x-axis $x(x-1)(x-2)=0$

$x=0$, $x=1$ and $x=2$.



* since areas are above and below the x-axis we must work them out separately.

$a_1 = \int_0^1 (x^3-3x^2+2x)\,dx$

$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1$

$= \left(\frac{1}{4}-1+1\right) - 0$

$= \frac{1}{4}$

$a_2 = -\int_1^2 (x^3-3x^2+2x)\,dx$

$= -\left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

$= -(4+8+4) + \left(\frac{1}{4}-1+1\right)$

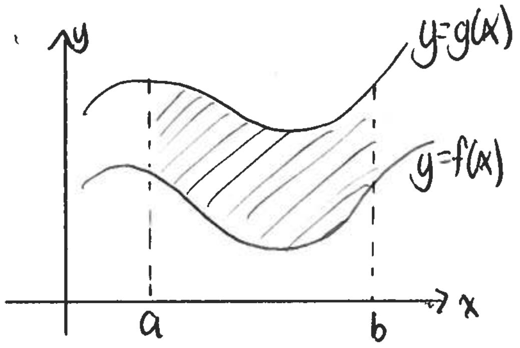
$= \frac{1}{4}$.

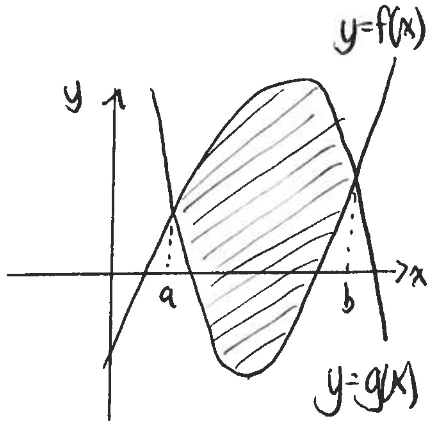
total area $= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$ square units.

Worksheet: 2(c) , 3, 11, 20, 22

Area Between Two Curves





$$\text{Area} = \int_a^b (upper - lower)dx$$
$$= \int_a^b (g(x) - f(x))dx$$

NB It doesn't matter if part of the area is below the x-axis for area between curves.

Example

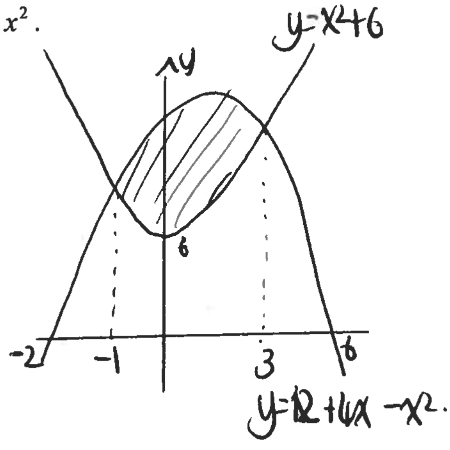
Calculate the area between the curves y = $x^2 + 6$ and y = $12 + 4x - x^2$.

Sketch. $+x^2$ ∪ $+6$ ↑

$y = 12 + 4x - x^2$ cuts x-axis $x^2 - 4x - 12 = 0$

$-x^2$ ∩ $(x-6)(x+2) = 0$

$x = 6 \quad x = -2$



Points of intersection $x^2 + 6 = 12 + 4x - x^2$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \quad x = -1$

area $= \int_{-1}^{3} (\text{upper} - \text{lower})\, dx$

$= \int_{-1}^{3} ((12 + 4x - x^2) - (x^2 + 6))\, dx$

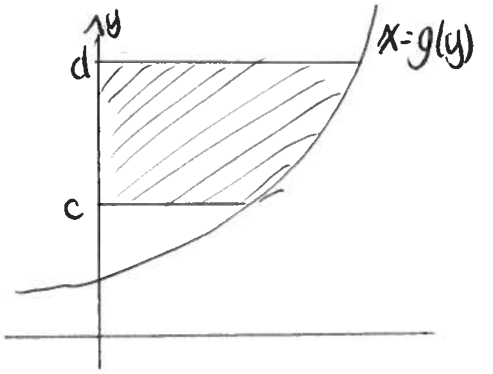
$= \int_{-1}^{3} (6 + 4x - 2x^2)\, dx$

$= [6x + 2x^2 - \frac{2}{3}x^3]_{-1}^{3} = (18 + 18 - 18) - (-6 + 2 + \frac{2}{3})$

Maths in Action Book 1 Page 90 Exercise 10b Questions 4 , 5 , 8

$= 21\frac{1}{3}$ square units.

Area Between the Curve and y-axis



note must write equation in the form
x = something in y's.

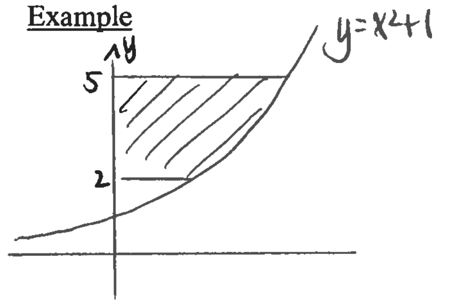
area = $\int_c^d g(y)\,dy$ ← integrate wrt y

limits for y

function in y's.

$$\text{Area} = \int_c^d g(y)dy$$

Example



Calculate the shaded area.

Here $y = x^2+1$

$x^2 = y-1$

$x = \sqrt{y-1}$

positive √ since using only this part of the graph.

area = $\int_2^5 \sqrt{y-1}\,dy$

$= \int_2^5 (y-1)^{\frac{1}{2}}\,dy$

$= \left[\frac{2}{3}(y-1)^{\frac{3}{2}} \right]_2^5$

$= \frac{2}{3}.4^{\frac{3}{2}} - \frac{2}{3}.1^{\frac{3}{2}}$

$= \frac{2}{3}\times 8 - \frac{2}{3}$

$= \frac{14}{3}$ square units.

NB areas to left of y-axis give negative value but correct magnitude.

Tips for Success....

- calculating areas <u>always</u> use a sketch.
  - → check if area above/below for between x-axis
    left/right for between y-axis
    bottom/top for between 2 curves.
- Remember → for areas between curve and <u>x-axis</u> everything in <u>terms of x</u>
  - → for area between curve and <u>y-axis</u> <u>everything in terms of y</u>

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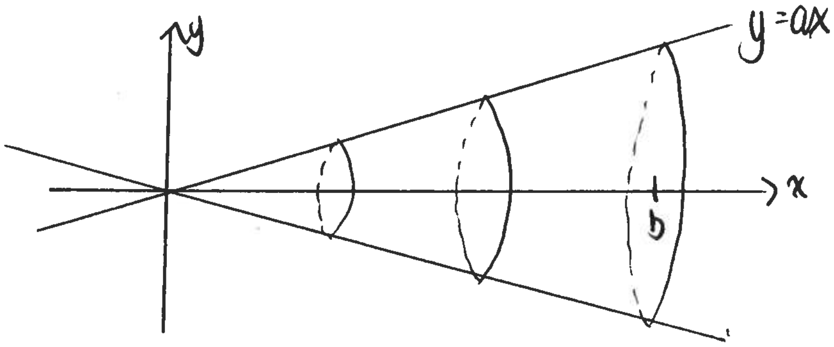
Worksheet Q15 - 18

Volumes of Revolution

Volumes of revolution are formed when a curve is rotated about the x-axis (or y-axis).
When a line is rotated a cone is formed.



$$V = \int_0^b \pi (ax)^2 \, dx$$

<u>Example</u>

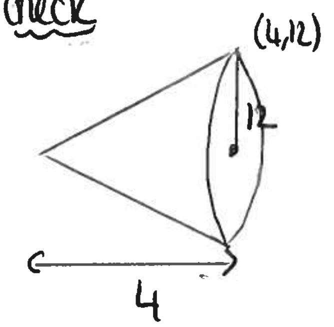
If $y = 3x$ and $b = 4$

$$V = \int_0^4 \pi (3x)^2 \, dx = 9\pi \left[\frac{x^3}{3} \right]_0^4$$

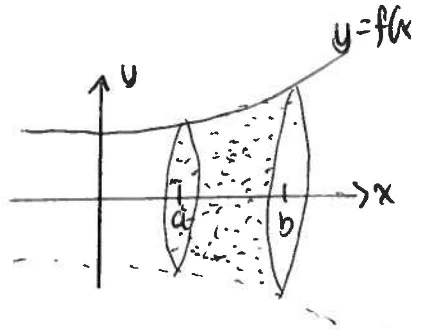
$$= 9\pi \frac{64}{3} - 0$$

$$= 192\pi \text{ units}^3$$

Check



$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 12^2 \times 4$$
$$= 192\pi \text{ units}^3$$



In general,

limits for $\underline{\underline{x}}$

LEARN

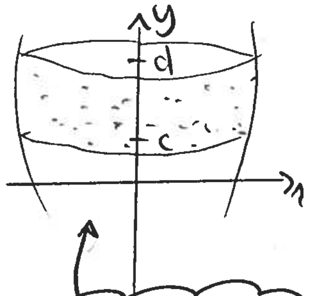
Using the $\underline{\underline{\text{x-axis}}}$ $\qquad V = \int_a^b \pi y^2 dx$

formula in terms of $\underline{\underline{x}}$

Using the $\underline{\underline{\text{y-axis}}}$ $\qquad V = \int_c^d \pi x^2 dy$

formula in terms of y

limits for y



Compare with $A = \pi r^2$. Integrals are summing up circles.

Example

Find the volume of revolution obtained between x = 1 and x = 3 when the curve $y = x^2 + 1$ is rotated (i) about the x-axis
(ii) about the y-axis.

(i) About x-axis → limits for x
equation in x's.
ie $y = x^2 + 1$.

$$V = \int_1^3 \pi (x^2+1)^2 dx$$
$$= \pi \int_1^3 (x^4 + 2x^2 + 1)\, dx$$
$$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_1^3$$
$$= \pi \left(\left(\frac{3^5}{5} + \frac{2 \times 3^3}{3} + 3 \right) - \left(\frac{1}{5} + \frac{2}{3} + 1 \right) \right)$$
$$= \frac{1016}{15}\pi \text{ units}^3$$

(ii) About y-axis - limits for y
equations in y's

Limits When $x=1$ $y=1^2+1=2$
$x=3$ $y=3^2+1=10$

Equation $y=x^2+1$
$x^2=y-1$
$x=(y-1)^{\frac{1}{2}}$
(x is positive $\Rightarrow +\sqrt{\ }$)

$$V=\int_2^{10}\pi\left((y-1)^{\frac{1}{2}}\right)^2 dy$$
$$=\pi\int_2^{10}(y-1)\,dy$$
$$=\pi\left[\frac{y^2}{2}-y\right]_2^{10}$$
$$=\pi\left((50-10)-(2-2)\right)$$
$$=40\pi \text{ units}^3$$

Tips for Success….

Remember

$V=\int_a^b \pi y^2\,dx$ — replace with x's
about x-axis
all in x's

$V=\int_c^d \pi x^2\,dy$ — replace with y's.
about y-axis
all in y's.

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Worksheet Q8, 10, 12, 13, 14