REVISION PACK

HIGHER GRADE

Higher Mathematics

Revision Pack

The Circle

1. State the centre and radius of each circle below :

   (a) $x^2 + y^2 = 100$ (b) $x^2 + y^2 + 6x - 10y - 2 = 0$ (c) $2x^2 + 2y^2 - 6x + 3y - \frac{1}{2} = 0$.

2. Find the equations of the circles specified as follows :

   (a) Centre (2,-4) , radius 5.

   (b) Centre (-1,3) and passing through the point (5,11).

   (c) Passing through the points A(3,7) and B(-5,1) and having AB as a diameter.

3. Find the equation of the circle passing through the points P(-5,1) , Q(-1,-1) and R(3,7).

4. Show that the following two circles touch externally at a single point :

   $x^2 + y^2 - 10x - 14y + 54 = 0$ *and* $x^2 + y^2 - 4x - 2y = 0$.

5. Show that the point E(-1,2) lies on the circumference of the circle with equation $x^2 + y^2 - 2x - 3y - 1 = 0$ and find the coordinates of the point F, given that EF is a diameter.

6. Verify that the point A(4,2) lies on the circumference of the circle $x^2 + y^2 + 3x - 3y - 26 = 0$ and find the equation of the tangent at this point.

7. A circle has as its equation $x^2 + y^2 - 4x - 10y + 19 = 0$.

   (a) Find the equation of the chord of this circle which has (1,3) as its mid-point.

   (b) Find the coordinates of the two points where this chord cuts the circle and hence verify that (1,3) is the mid-point of this chord.

8. A circle has as its equation $x^2 + y^2 - 6x + 16y + 8 = 0$.

   (a) Verify that the point (4,0) lies on the circle.

   (b) Find the equation of the tangent at this point.

   (c) Find the equation of the parallel tangent.

The Straight Line

1. Find the gradients of the lines between the following sets of points :

   (a) A(2,8) and B(-4,10) (b) P(-3,-6) and Q(-1,2) .

2. The points E and F have coordinates (2,-5) and (-4, a) respectively.
   Given that the gradient of the line EF is $\frac{2}{3}$, find the value of a .

3. If the points (3 , 2) , (-1 , 0) and (4 , k) are collinear, find k .

4. Find the equations of the lines specified as follows :

   (a) Passing through the point P(2,-3) with gradient 4 .

   (b) Passing through the points A(-1,1) and B(3,-1) .

   (c) Passing through (4,-5) and *parallel* to the line with equation $3x + 2y = 8$.

5. Prove that the lines with equations $y = -5x + 2$ and $2x - 10y = 15$ are *perpendicular.*

6. The line PQ has end-points P(6,-5) and Q(-2,-3) . Find the equation of the *perpendicular bisector* of PQ.

7. Triangle ABC has vertices (2,6) , (-7,4) and (4,-8) respectively.

   (a) Find the equation of the *median* from B to AC .

   (b) Find the equation of the altitude from A to BC .

8. Triangle PQR has vertices (2,3) , (-3,-2) and (3,0) respectively.

   (a) Find the equations of the *perpendicular bisectors* of sides RQ and PR .

   (b) Find the coordinates of the point T , the point of intersection of these two bisectors.

   (c) Show that P , T and Q are collinear.

9. ABCD is a parallelogram whose diagonals intersect at E . If A , B and E are the points (-1,0) , (3,-2) and (1,4) respectively, find the equation of DC .

Functions

1. The functions $f(x) = x^2 + x$ and $g(x) = 3x - 1$ are defined on the set of positive real numbers (R^+).

   (*a*) Evaluate *i)* $g(f(2))$ *ii)* $f(g(-1))$.

   (*b*) Find a formula for *i)* $f(g(x))$

   ii) $g(f(a))$

   iii) $f(f(k))$

   (*c*) Find a formula for the inverse function $g^{-1}(x)$.

2. On a suitable set of real numbers, functions f and h are defined by :

$$f(x) = \frac{2}{3x-3} \quad \text{and} \quad h(x) = \frac{1}{x^2} + 1$$

   (*a*) Find $f(h(x))$ in its simplest form.

   (*b*) Find a formula for the function $f^{-1}(x)$ and state a suitable domain for this inverse function.

3. A function is defined as $f(x) = \dfrac{1}{2x+1}$.

   (*a*) Define the largest possible domain available to this function on the set of real numbers.

   (*b*) Show that the inverse function $f^{-1}(x)$ can be written in the form $f^{-1}(x) = \dfrac{1-x}{2x}$.

   (*c*) Hence find the two possible values of x such that $f^{-1}(x) = f(x)$.

4. Two functions f and h are defined on the set of real numbers as follows :

$$f(x) = 2x + 1 \quad , \quad h(x) = \frac{1}{x-1}, x \neq 1 .$$

   (*a*) Given that $g(x) = f(h(x))$ show that $g(x)$ can be written as $g(x) = \dfrac{x+1}{x-1}$.

   (*b*) Hence verify that $g^{-1}(x) = g(x)$.

Differentiation 1

1. Differentiate each of the following with respect to x :

(a) $3x^2 + 6x$ (b) $x^4 - 2x^{-1} + 4$ (c) $(2x+3)^2$

2. Differentiate each of the following functions with respect to the relevant variable :

(a) $f(x) = x^3(x - x^2)$ (b) $g(x) = 3x^2 + \dfrac{1}{x^3}$ (c) $h(x) = \dfrac{1}{x} + \dfrac{1}{3x^2}$

(d) $f(t) = t^{1/2}(t^2 + t^{3/2})$ (e) $g(p) = \dfrac{1}{p}(p^{-1/3} - p^{2/3})$ (f) $h(u) = \sqrt{u} + \dfrac{1}{2\sqrt{u}}$

(g) $g(x) = \dfrac{x^5 + 2x^2}{x^4}$ (h) $f(v) = \dfrac{5v+2}{\sqrt{v}}$ (i) $h(t) = \dfrac{1}{\sqrt{t}}\left(\dfrac{t^2 + t^{3/2}}{t}\right)$

3. Calculate the rate of change of :

(a) $f(x) = x^3 - 2x^2$ at $x = 2$ (b) $h(t) = (t-1)(2t+3)$ at $t = -1$

4. The amount of pressure P (pounds per square inch) within a cylinder varies with time t (milliseconds) according to the formula $P(t) = 400t - 50t^2$.

(a) Calculate the rate of change of P when $t = 2$.
(b) Calculate how fast P is changing when $t = 4$.
(c) How is the rate of P changing when $t = 5$? Comment.

5. (a) Find the equation of the tangent to the curve with equation $y = 3x^2 - 2x$ at the point where $x = -1$.

(b) Find the equation of the tangent to the curve $y = \dfrac{1}{x} + x$ at the point where $x = \frac{1}{2}$.

6. A function f is given by $f(x) = 3x^2 - 2x^3$. Determine the interval on which f is increasing.

7. Find the stationary points of the curve $y = x^2(12 - 4x - 3x^2)$ and determine their nature.

8. A curve has as its equation $y = x(2-x)^2$.

(a) Find the stationary points of the curve and determine the nature of each.

(b) Write down the coordinates of the points where the curve meets the coordinate axes.

(c) Sketch the curve.

Sequences and Recurrence Relations

1. (a) If £300 is invested at an interest rate of $4 \cdot 5\%$ per annum, find the total value of the investment after 5 years.

   (b) How many years, at the same rate of interest, would it take to double the investment ?

2. A recurrence relationship is defined as $U_{n+1} = aU_n + b$.

   (a) List the four terms U_1 to U_4 of this sequence when $U_0 = 30$, $a = 0 \cdot 8$ and $b = 4$.

   (b) Describe why this particular sequence has a limit.

   (c) Calculate the limit of this sequence.

3. A sequence is defined by the recurrence relation $U_{n+1} = kU_n + c$, where k and c are constants.

   (a) Given that $U_0 = 60$, $U_1 = 10$ and $U_2 = -15$, form a system of equations and solve it to find k and c .

   (b) Find U_4 .

4. An unstable atomic particle decays in mass by half every two hours. At the end of each 2 hour period it is bombarded by atomic matter which allows it to recover a mass of 12 a.mu.

   (a) By considering a suitable recurrence relationship and taking the original mass of the particle to be 40 a.m.u. , calculate the number of hours it will take to decay to a value which is below 32% of its original mass.

   (b) If the mass of this particle falls below $12 \cdot 3$ a.m.u. it becomes highly unstable resulting in an explosion.
   Should the scientists allow this experiment to continue over a long period of time ?
   Explain your answer.

5. An antibiotic in the body dissipates at the rate of 8% during each 10 minute period from the initial administration.

   (a) If an initial dose of 100 units is given what amount of antibiotic will still be present in a patients bloodstream after 1 hour ?

   (b) A doctor prescribes this drug to a patient. It is known that in order to be effective a level of at least 100 units must be maintained over the long term, however the drug becomes toxic if the level exceeds 300 units.
   The doctor prescribes 100 units every hour. Is this treatment satisfactory ?
   Your answer must be accompanied by the appropriate working and justifications.

Differentiation 2

1. Differentiate, expressing your answers with positive indices :

(a) $(3x+2)^4$ (b) $(x^2-4)^{3/2}$ (c) $\sqrt{(5-4x)}$

(d) $\dfrac{1}{(x^2+3x)^3}$ (e) $\dfrac{-4}{x-2}$ (f) $\dfrac{2}{\sqrt{(x^2-7)}}$

2. Differentiate, expressing your answers with positive indices :

(a) $2\sin 3x$ (b) $\cos^3 x$ (c) $5\sin^2 x$

(d) $\cos(3-2x)$ (e) $\dfrac{1}{\sin x}$ (f) $\dfrac{1}{\sqrt{\cos x}}$

3. A curve has as its equation $y = \dfrac{5}{4x+1}$, where $x \neq -\frac{1}{4}$.

Find the equation of the tangent to this curve at the point where $x = 1$.

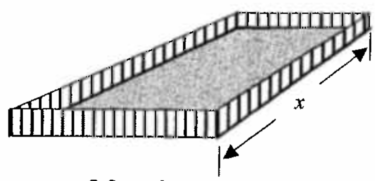
4. A curve has as its equation $y = 2\sin\frac{1}{2}\theta$ *for* $0 \leq \theta < \pi$.

(a) Find the exact value of y when $\theta = \frac{2}{3}\pi$.

(b) Find $\dfrac{dy}{d\theta}$ and evaluate it for $\theta = \frac{2}{3}\pi$.

(c) Hence show that the equation of the tangent to the curve at the point where $\theta = \frac{2}{3}\pi$ can be written in the form $2y = \theta + (2\sqrt{3} - \frac{2}{3}\pi)$.

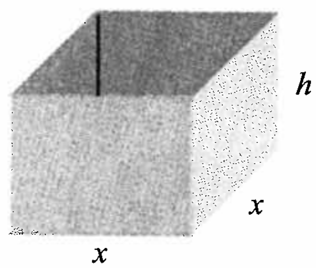
5. A rectangular field is x metres long. It is enclosed by 200 metres of fencing.



(a) Find an expression in terms of x for the breadth of the field.

(b) Hence find the greatest area which can be enclosed with the 200 metres of fencing.

6. An open cistern with a square base and vertical sides is to have a capacity of 4000 cubic feet.



(a) Taking the length of the square base to be x feet, find an expression for the height h in terms of x.

(b) Hence show that the surface area, A square feet, of the cistern can be written in the form

$$A(x) = x^2 + \frac{16000}{x}$$

(c) Find the dimensions of the cistern so that the cost of cladding it in lead sheet will be a minimum.

Quadratic Theory

1. Write each of the following quadratic expressions in the form $a(x+b)^2 + c$:

(a) x^2+6x-3 (b) x^2-5x+1 (c) $3x^2+12x+2$

(d) $2x^2-6x-4$ (e) $4+8x-x^2$ (f) $1-4x-2x^2$

2. (a) Show that the function $f(x) = 2x^2 - 16x + 7$ can be written in the form $f(x) = a(x+p)^2 + q$ and write down the values of a , p and q.

Hence state the minimum value of the function and the corresponding value of x .

(b) Express the function $g(\alpha) = \sin^2\alpha - \sin\alpha - 1$ in the form $g(\alpha) = (\sin\alpha + p)^2 + q$ and write down the values of p and q .

Given that $0 < \alpha < \pi/2$, state the minimum value of g and the corresponding replacement for α .

3. (a) Find the value of k which results in the equation $kx^2 + 2kx - 1 = 0$ having **equal** roots, given that $k \neq 0$?

(b) A quadratic equation is given as $x^2 + (p-3)x + (1/4 - 3p) = 0$.

For what values of p will the above equation have i) equal roots ;
ii) no real roots.

(c) Show that the roots of the equation $(t-1)x^2 + 2tx + 4 = 0$ are real for all values of t .

4. A computer generating random products using two variables delivers out the following four expressions on printed cards :

card 1: $4x^2$

card 2: $2 - k$

card 3: $2k+1$

card 4: $4x+1$

(a) Given that the product of cards 1 and 3 is equal to the product of cards 2 and 4 , show that the following equation can be constructed

$$(8k+4)x^2 + (4k-8)x + (k-2) = 0$$

(b) Hence find the values of k so that the above equation has equal roots.

Systems of Equations

1. Solve each of the following systems of equations :

(a) $x^2 + y^2 = 29$
$x - y = 3$

(b) $x + 2y = 16$
$x^2 - 7xy + 4y^2 + 74 = 0$

(c) $x - 2y = -1$
$x^2 - 2xy + 4y^2 = 157$

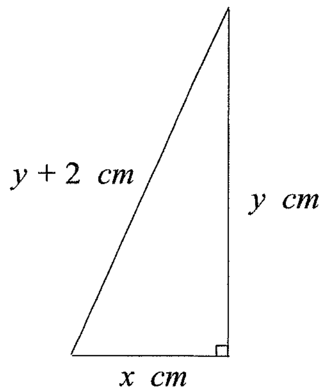
(d) $y + 4 = 3x$
$2x^2 + xy + 2x = 3$

2. Solve :

(a) $7x^2 - 12x - 8y = 11$
$3x - 4y = 1$

(b) $5x - 2y + 3 = 0$
$25x^2 - 5xy + y^2 = 21$

3. The hypotenuse of a right-angled triangle is 2cm more than the next shortest side as shown in the diagram.
The perimeter of the triangle is 60 centimetres.

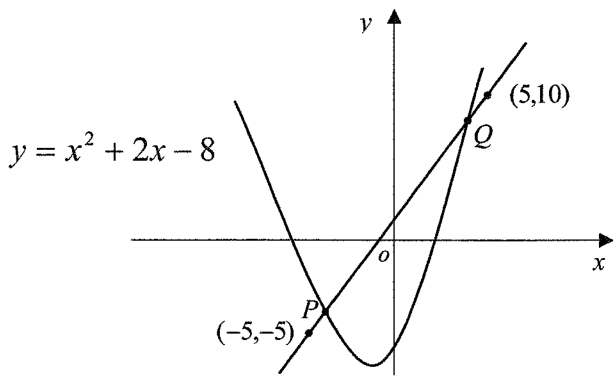


(a) Use pythagoras to form an equation connecting the lengths of the sides.

(b) Form a second equation involving the perimeter of the triangle.

(c) Hence solve a system of equations to find the lengths of the three sides of the triangle.

4. The line through the points $(-5,-5)$ and $(5,10)$ meets the curve with equation $y = x^2 + 2x - 8$ at the points P and Q as shown in the diagram below.



Establish the coordinates of P and Q .

Vectors (1)

1. The vertices of a triangle are A(2,3) , B(-4,1) and C(5,-4).

   (a) Find in component form the vectors represented by the following displacements :

   i) $\overrightarrow{AC}$ *ii)* $\overrightarrow{CB}$

   (b) Calculate the magnitude of $\overrightarrow{AB}$.

2. Given that $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix}$.

   (a) Find in component form the vectors represented by i) $\underset{\sim}{a} + 2\underset{\sim}{b}$ ii) $3\underset{\sim}{c} - \underset{\sim}{a}$.

   (b) Calculate $\left|(2\underset{\sim}{a} - \underset{\sim}{b})\right|$.

3. Two forces are represented by the vectors $F_1 = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $F_2 = \underset{\sim}{i} + 4\underset{\sim}{k}$.
   Find the magnitude of the resultant force $F_1 + F_2$.

4. Points P , Q and R have coordinates (1,3,2) , (-2,0,6) and (-4,1,-5) respectively.

   Given that $2PQ = RS$, find the coordinates of the point S.

5. Prove that the points E(2,-2,0) , F(1,4,-3) and G(-1,16,-9) are collinear, and find the ratio $EF : FG$.

6. Two vectors are defined as $V_1 = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $V_2 = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$.
   Show that these two vectors are perpendicular.

7. Vector $\underset{\sim}{v}$ is a *unit vector* parallel to the vector $\underset{\sim}{u} = \begin{pmatrix} \sqrt{2} \\ -1 \\ 1 \end{pmatrix}$. Find $\underset{\sim}{v}$.

8. The magnitude of vector $\underset{\sim}{a}$ is 2 *units* and the magnitude of vector $\underset{\sim}{b}$ is $2\sqrt{2}$ *units*.
   Given that the angle between the two vectors is $45°$ evaluate the scalar product $\underset{\sim}{a} \cdot (\underset{\sim}{a} - \underset{\sim}{b})$
   What can you say about the angle between the vectors $\underset{\sim}{a}$ *and* $\underset{\sim}{a} - \underset{\sim}{b}$?

Vectors (2)

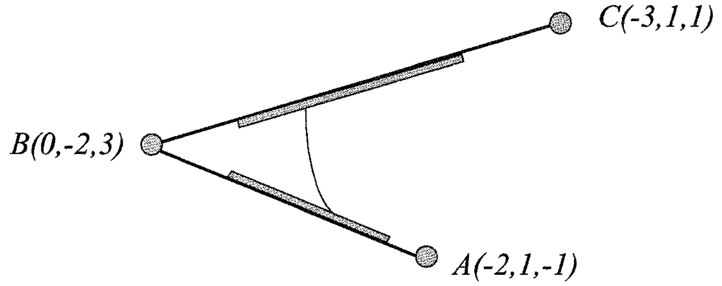
1. Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are given as $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$ respectively.

(a) Prove that the angle between these two vectors is obtuse.

(b) Vector $\underset{\sim}{u} = 2\underset{\sim}{a}$ and $\underset{\sim}{v} = \underset{\sim}{a} + \underset{\sim}{b}$. Calculate the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(c) Vector $\underset{\sim}{c}$ has components $\begin{pmatrix} 2 \\ k \\ 5 \end{pmatrix}$. Find k, given that $\underset{\sim}{c}$ is perpendicular to $\underset{\sim}{a}$.

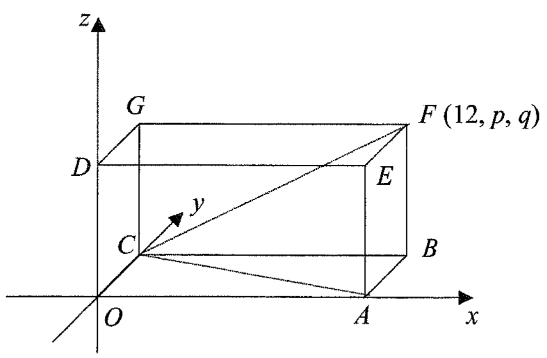
2. The diagram below is a schematic of an elastic coupling used as part of a cassette door mechanism. Relative to an origin O , points A , B and C have coordinates $(-2,1,-1)$, $(0,-2,3)$ and $(-3,1,1)$ respectively .



Calculate the size of angle ABC .

3. A cuboid is placed on coordinate axes as shown.
The dimensions of the cuboid are in the ratio $OA : AB : BF = 4 : 1 : 2$

The point F has coordinates $(12, p, q)$ as shown.



(a) Establish the values of p and q .

(b) Write down the coordinates of the points A and C and hence calculate the size of $\angle ACF$.

Integration

1. Find : a) $\int (9x^2 + 6x)\, dx$ b) $\int (3x - x^2)^2\, dx$ c) $\int \frac{(\sqrt{x} + 1)}{x^2}\, dx$

2. Evaluate each of the following definite integrals :

(a) $\int_1^4 \frac{x^2 + 2x}{\sqrt{x}}\, dx$ (b) $\int_{-1}^{1} \frac{1}{v}\left(3v^3 - v^{-1/3}\right) dv$

3. A curve is such that its derivative is defined as $\frac{dy}{dx} = x - \frac{1}{x^2}$.
If the curve passes through the point $(2\ ,\ -\frac{5}{2})$, find the equation of the curve.

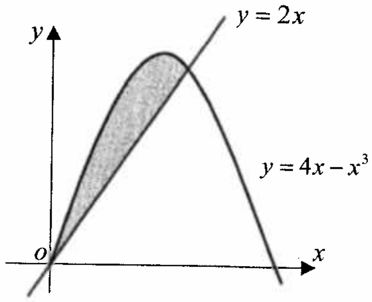
4. For the curve $y = f(x)$ where $x > 0$, $\frac{dy}{dx} = 6 - \frac{24}{x^2}$

If the minimum turning value of the function is 20 , find $f(x)$.

5. Find the area enclosed between the parabola $y = 4x - x^2$ and the straight line $y = x$.

6. (a) Find the area in the first quadrant bounded by the curve $y = 4x - x^3$ and the x - axis .

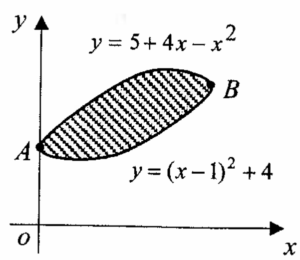
(b) Show that the line $y = 2x$ divides this area into two parts which are in the ratio 1 : 3 .



7. The cross-sectional area of a ships hydro-foil is shown in the diagram opposite with rectangular axes having been added. The top surface has as its equation $y = 5 + 4x - x^2$ and the lower surface $y = (x-1)^2 + 4$.

(a) Establish the coordinates of A and B , the points of intersection of the two curves.

(b) Hence calculate the cross-sectional area of the hydro-foil in square units.



Polynomials

1. Use synthetic division to evaluate each of the following :

   (*a*) Given that $f(x) = 3x^4 - x^3 + 2x^2 - 6$, find $f(2)$ and $f(-1)$.

   (*b*) Given that $y = x^5 + 2x^3 - 4x + 2$, find y when $x = -1$.

2. Find the quotient and remainder when dividing :

   (*a*) $x^3 + 7x^2 + 4x - 1$ by $x + 1$

   (*b*) $x^3 - 2x^2 + 3x - 10$ by $x - 2$

   (*c*) $2x^3 - x^2 - 3x - 1$ by $2x + 1$

3. Prove that $x + 2$ is a factor of $x^3 - x^2 - 10x - 8$ and hence find the other factors.

4. Prove that $x - 3$ is a factor of $x^3 - x^2 - 9x + 9$ and hence fully factorise the expression.

5. Factorise fully : (*a*) $x^3 - 21x + 20$ (*b*) $4x^3 - 8x^2 + x + 3$

6. (*a*) Find the value of k so that $x^3 + 5x^2 - 4x + k$ is exactly divisible by $x - 2$.

   (*b*) For what value of c is $x - 1$ a factor of $x^3 + cx^2 - 5x + 6$?

   (c) For what values of a and b are $x + 2$ and $2x - 1$ both factors of $4x^3 + ax + b$?

7. Solve the equations : (*a*) $3x^3 - 7x^2 + 4 = 0$

   (*b*) $x^3 = 7x + 6$

8. Prove that $x + 2y$ is a factor of $x^4 + 10xy^3 + 4y^4$.

9. The package below is designed in such a way that its length is 3 *cm* more than its height and its breadth is 1 *cm* more than its height.

   (*a*) Taking the height of the package as h , establish a formula for the volume of the package in terms of h .

   (*b*) Given now that the volume of the package is 300 cm^3, form an equation and show that the height of the package must lie between 5 *cm* and 6 *cm* .

   (*c*) Hence approximate the height to 1 decimal place.

Trig. Equations

1. Solve each of the following equations :

(a) $\sqrt{3}\tan x^\circ + 2 = 1$, for $0 \le x < 360^\circ$

(b) $4\sin^2\theta = 1$, for $0 \le \theta < 2\pi$

(c) $2\sin 2x^\circ + 1 = 0$, for $0 \le x < 360^\circ$

(d) $3\cos 3x^\circ - 1 = 0$, for $0 \le x < 180^\circ$

2. Solve each of the following equations for $0 \le x < 360^\circ$:

(a) $3\sin^2 x^\circ + 2\sin x^\circ - 1 = 0$ (b) $2\cos 2x^\circ - 3\sin x^\circ - 1 = 0$

(c) $\cos x^\circ = \frac{1}{2}\sin 2x^\circ$ (d) $\cos 2x^\circ - \sin x^\circ = 4\sin^2 x^\circ$

3. Solve the equation $8\cos x^\circ + 15\sin x^\circ = 10$ for $0 \le x < 360^\circ$.

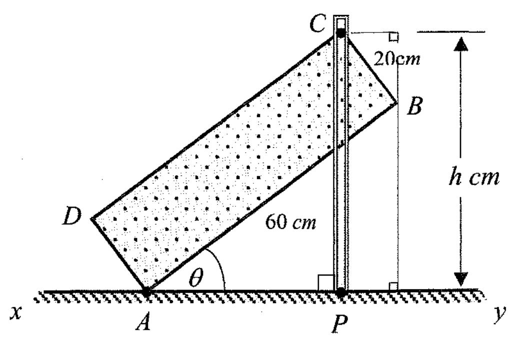
4. Solve $\sqrt{3}\sin\alpha - 1 = \cos\alpha$ for $0 \le \alpha < 2\pi$.

5. Solve the equation $4\sin 2x^\circ - 3\cos 2x^\circ = 1$ *for* $0 \le x < 180^\circ$.

6. (a) Express $4\cos^2 x^\circ + 4\sin x^\circ\cos x^\circ - 3 = 0$ in the form $P\cos 2x^\circ + Q\sin 2x^\circ = R$, and write down the values of P , Q and R .

(b) Hence solve the equation $4\cos^2 x^\circ + 4\sin x^\circ\cos x^\circ - 3 = 0$, for $0 \le x < 90^\circ$.

7. The diagram opposite shows a rectangular rocket launcher $ABCD$. The launcher is hinged at point A. The sliding rod PC has been fixed such that $\angle APC = 90^\circ$. $AB = 60\ cm$ and $BC = 20\ cm$ and $\angle BAY = \theta$ as shown.



(a) Use the diagram to show that the height of C above XY can be expressed as

$$h = 60\sin\theta + 20\cos\theta$$

(b) Given that $h = 54\ cm$, claculate the size of angle θ to the nearest degree.

The Exponential & Logarithmic Function

1. Solve each of the following equations for x, rounding your answers to two decimal places.

   (*a*) $7^x = 24$ (*b*) $3^{-2x} = \frac{1}{50}$

2. The rate of decomposition of an acid in a solution obeys the law $C = 4e^{-0\cdot 025t}$, where C is the concentration in millilitres of acid left after t minutes.

   (*a*) What is the intial concentration of acid ?

   (*b*) Determine how long it takes for the concentration to reach 3ml, giving your answer to the nearest second.

   (*c*) How long does it take to reduce to *half* its original concentration ?

3. Solve for x in $2\log_2 \frac{1}{x} + 3\log_2 x = 4$.

4. Radioactive isotopes are used as indicators for bearing wear in high speed machinery.
   The rate of decay of these isotopes is measured in terms of their *half-life*, i.e. the time required for one half of the material to decay.
   For a particular isotope the mass present at any time M a.m.u. (atomic mass units), at time t hours is given by :
   $$M = M_0 e^{-kt}$$, where k is a constant and M_0 is the initial mass.

   Experiments have shown that at $t = 3$, $M = 0\cdot 8M_0$.

   (*a*) Find the value of k correct to 3 decimal places.

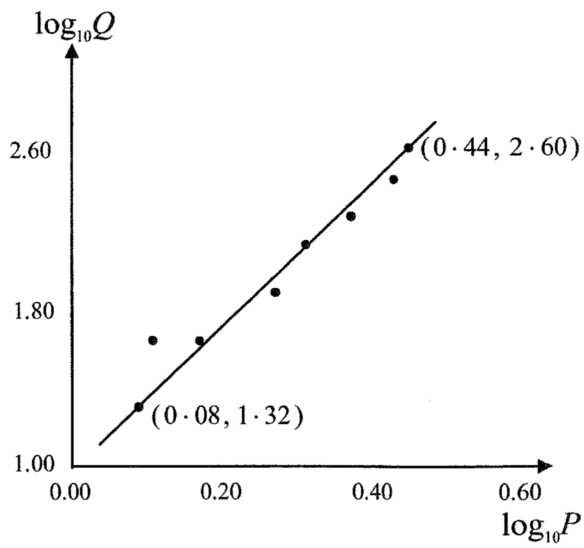
   (*b*) Hence evaluate the *half-life* of the isotope to the nearest minute.

5. From an experiment corresponding replacements for P and Q were found.

   $\log_{10} P$ was plotted against $\log_{10} Q$ and a best fitting straight line drawn, as shown in the diagram.

   (*a*) Show that a possible relationship between P and Q is of the form :
   $$Q = kP^n \ .$$

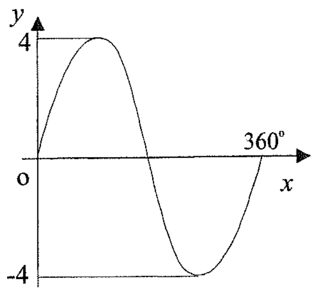
   (*b*) Find the values of n and k to one decimal place.



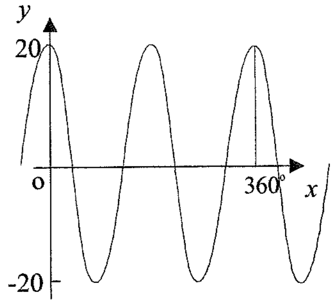
Graphicacy

1. Write down an equation which describes the relationship between x and y in each graph below :

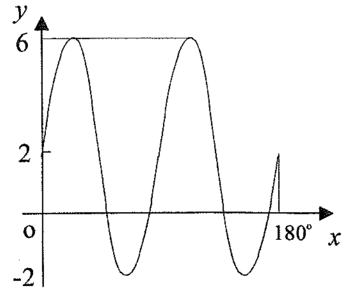
(a)



(b)



(c)



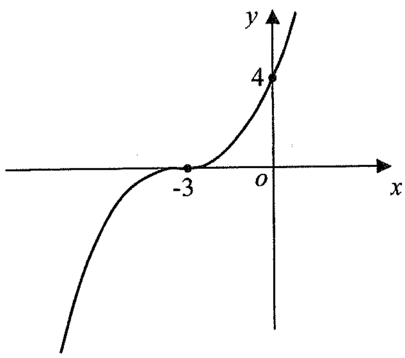
2. The graph of $y = f(x)$ is shown opposite.

(a) Draw a sketch of $y = -f(x)$.

(b) Draw a sketch of $y = f(-x) + 1$.

(c) Draw a sketch of $y = f(x-3)$.

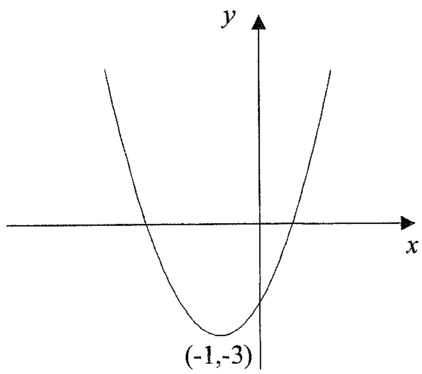
(d) Draw a sketch of $y = f^{-1}(x)$.



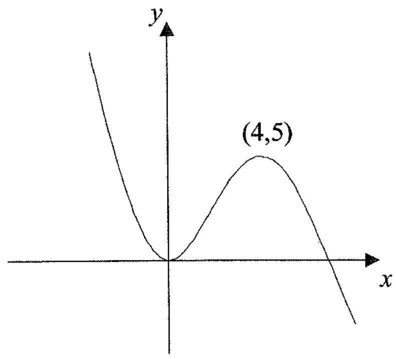
3. For each graph below $y = f(x)$

Draw a sketch of the graph of the derivative, $y = f'(x)$, of each function.

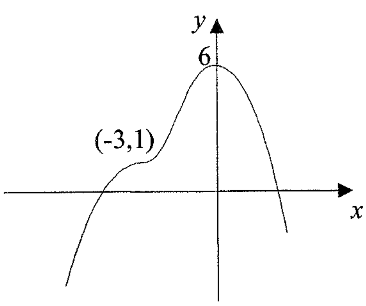
(a)



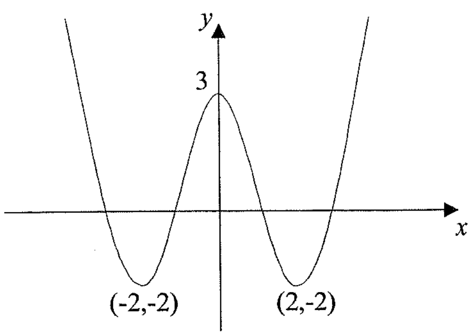
(b)



(c)



(d)



The Wave Function ($a\cos x + b\sin x$)

1. (a) Express $\sqrt{3}\cos x^\circ + \sin x^\circ$ in the form $k\sin(x+\alpha)^\circ$, where k and α are constants and $k > 0$.

   (b) Hence state the minimum value of the function f given that
   $f(x) = \dfrac{12}{\sqrt{3}\cos x^\circ + \sin x^\circ + 4}$, and the corresponding replacement for x.

2. The formula $h = 3\sin 15t^\circ + 4\cos 15t^\circ + 25$ represents the height, in metres, of a wave-power boom above the sea bed at time t hours after midnight.

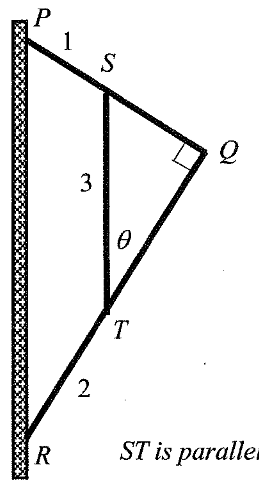
   (a) Express h in the form $k\cos(15t - \alpha)^\circ + 25$.

   (b) What is the booms minimum height above the sea bed and at what time does this minimum clearance occur ?

   (c) Sketch the graph of h against t for $0 \le t \le 24$, showing clearly all relevant points.

   (d) The boom generates the most power when it is 27 metres or more above the sea bed. For what length of time is this ideal situation in operation ? Give your answer correct to the nearest minute.

3. The diagram opposite shows an A-Frame used to support an inspection platform on an North Sea oil-rig. Angle SQT is a right-angle, $\angle QTS = \theta$ *radians* and all lengths are in metres.



ST is parallel to PR

   (a) Show that the length of RQ is given as $2 + 3\cos\theta$ and PQ as $1 + 3\sin\theta$.

   (b) Hence express PR^2 in the form $a\cos\theta + b\sin\theta + c$ and write down the values of a, b and c.

   (c) Find the maximum length of PR and the corresponding value of θ. Give your answers correct to 1 d.p.

4. (a) Show that $4\sin(x+30)^\circ - \cos x^\circ$ can be written as $2\sqrt{3}\sin x^\circ + \cos x^\circ$.

   (b) Express $2\sqrt{3}\sin x^\circ + \cos x^\circ$ in the form $k\cos(x-\alpha)^\circ$, where $k > 0$ and α is acute, and write down the values of k and α.

   (c) Hence solve the equation $4\sin(x+30)^\circ - \cos x^\circ = 1$ for $0 \le x < 360$.

Mixed Exercise (1)

1. (a) Verify that $x+3$ is a factor of $2x^3+3x^2-11x-6$ and hence factorise the expression completely.

   (b) Find the equation of the tangent to the curve $y=8x+\dfrac{1}{x}$ at the point where $x=\frac{1}{2}$.

2. (a) Find the equation of the tangent to the circle $x^2+y^2+2x-4y-5=0$ at the point $(2,3)$ on the circle.

   (b) Show that the line $y=2x-8$ is a tangent to the circle with equation $x^2+y^2-4x-2y=0$, and find the coordinates of the point of contact.

3. (a) Given that $v=\sqrt{(2x+1)}$, find

   i) $\dfrac{dv}{dx}$ ii) x given that $\dfrac{dv}{dx}=5$.

   (b) Given that $f(x)=\sin 4x-2\cos^2 x$ evaluate $f'(\frac{\pi}{4})$.

4. Triangle ABC has vertices (-2,5) , (6,1) and (2,-7) respectively.

   (a) Find the equation of the perpendicular bisector of BC.

   (b) Show that this perpendicular passes through the mid-point of side AC.

   (c) The perpendicular produced meets the line through A with gradient $\frac{3}{4}$ at T , find the coordinates of T .

5. A and B are the points with coordinates (3,0,1) and (4,0,2) respectively.

   (a) Given that $\vec{BP}=k\vec{AB}$, establish the coordinates of P in terms of k .

   (b) Given now that OP is perpendicular to AB, find the value of k .

6. For what values of k does the quadratic equation $kx^2+2x+(6k-1)=0$, have equal roots ?

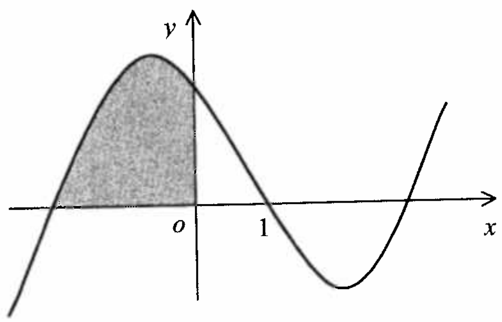
Mixed Exercise (2)

1. A , B and C are the points $(\frac{7}{2}, -1, -2)$, $(2, 1, -3)$ *and* $(\frac{13}{2}, -5, 0)$.

   (a) Show that A , B and C are collinear and state the value of the ratio BA : AC.

   (b) If the point $P(k, 1, 1)$ is such that BP is perpendicular to AC, find the value of k .

2. The diagram below shows a sketch of the graph of $y = f(x)$ where $f(x) = x^3 - 2x^2 - 5x + 6$.

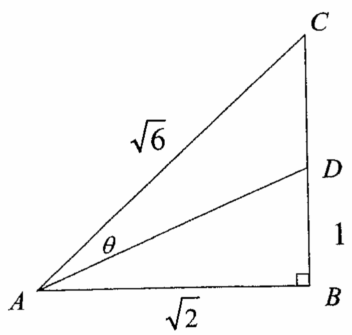


   (a) Show that $x - 1$ is a factor of this function and hence fully factorise $f(x)$.

   (b) Calculate the shaded area.

3. In the diagram opposite, show that the exact value of

$$\cos\theta = \frac{4}{3\sqrt{2}}$$



4. Find the equation of the tangent to the curve $y = 3x^2 - 2$ at the point where $x = 2$.

5. Solve the equation $2\cos 2x° = \sin x° - 1$, for $0 \le x < 360$.

6. A competitor in a long distance yacht race loses his mast in a storm. Now drifting helplessly he decides to ration his food supply. The only food on board is 16 kg of salted fish. He decides to eat 10% of his remaining food each day and is confident he can catch one fish each night (on his solitary hook) weighing an average of $0{\cdot}6$ kg.

   (a) How long is it before his supply drops below 12 kg ?

   (b) Will he ever run out of food ? Discuss the limit of this sequence.

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Mixed Exercise (3)

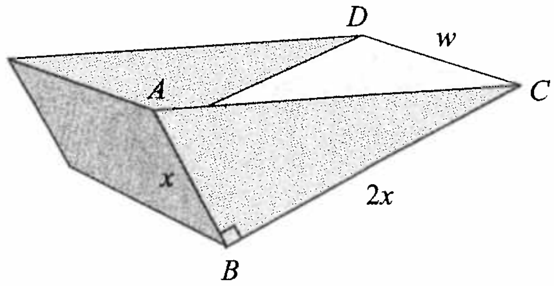
1. The functions $f(x) = 4 - x^2$ and $g(x) = 2k - x$, where k is a constant , are defined on the set of *positive real numbers*.

   (a) Show that $f(g(x)) + 2\left[g(f(x)) + 4k\right]$ can be written as $x^2 + 4kx - 4(k^2 - 3k + 1)$.

   (b) Hence find the values of k such that the equation $f(g(x)) + 2\left[g(f(x)) + 4k\right] = 0$ has equal roots.

2. Find (a) $\displaystyle\int \frac{x^3 - \sqrt{x}}{x^2}\,dx$ (b) $\dfrac{dy}{dx}$ when $y = \left(x^2 + \dfrac{1}{x}\right)^2$.

3. Tin sheeting is bent and sealed to form a feeding trough in the shape of the prism opposite.
Angle ABC is a right-angle.
The total amount of tin plate used is $6\frac{1}{2}$ square metres.
$AB = x$, $BC = 2x$ and $CD = w$.



   (a) Show that the surface area, A, in terms of x and w can be written as $A = 2x^2 + 3xw$.

   (b) Hence show that $w = \dfrac{13}{6x} - \dfrac{2x}{3}$.

   (c) If the volume of the trough is given as $V\ m^3$, show that $V(x) = \dfrac{13x}{6} - \dfrac{2x^3}{3}$.

   (d) Hence find the values of x and w for maximum volume.
   Give your answers correct to 2 decimal places

4. The points A , B and C have coordinates (2,1,1) , (-1,0,2) and (3 , -4 , -1) respectively.

   Calculate the size of $\angle ABC$.

5. A function is defined as $f(x) = x^3 - 3x + 2$, for $x \in R$.

   (a) Show that $x = -2$ is a root of the equation $x^3 - 3x + 2 = 0$ and hence find the other real root..

   (b) Find the coordinates of the stationary points of the graph $y = f(x)$ and determine their natures.

   (c) Sketch the graph of $y = f(x)$ marking clearly all relevant points.

Mixed Exercise (4)

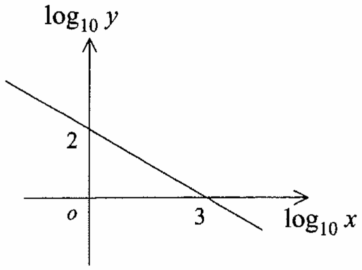
1. (a) Express $2x^2 - 12x + 1$ in the form $a(x+b)^2 + c$ and write down the values of a , b and c .

   (b) Express $3 + 5x - x^2$ in the form $p - (x+q)^2$ and write down the values of p and q .

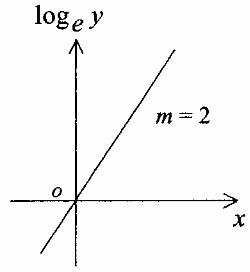
2. Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $W\cos(\theta - \alpha)$, where W is a constant and $0 \le \alpha < 2\pi$.

3. Write down a formula connecting x and y in each diagram below :

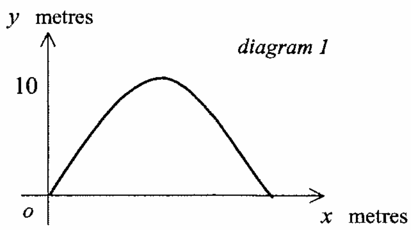
   (a)



   (b)

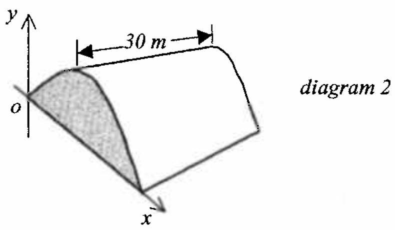


4. The cross-sectional area of a burial mound was discovered to be approximately parabolic as shown in *diagram 1.*



   (a) From the information supplied on the diagram, find a formula for y in terms of x for this curve.

   (b) Calculate the volume of earth in the mound , in cubic metres, given that the mound is a prism as shown in *diagram 2.*



5. Due to tidal variations the depth of water in a harbour, t hours after midnight, is d metres where

$$d = 8 + 3\sin(\tfrac{1}{2}t - 1)$$

   (a) What is the greatest and least depth of water in the harbour ?

   (b) When is high tide in the afternoon ?

   (c) How many hours elapse between successive high tides ?

6. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points (2,2) , (2,1) and (3,-1).

   (a) Write down three equations which must be satisfied by g , f and c .

   (b) Solve the equations as a system and hence establish the equation of the circle.

The Circle (answers)

1. (a) origin , $r = 10$ (b) C(-3,5) , $r = 6$ (c) $C(\tfrac{3}{2}, -\tfrac{3}{4})$, $r = \tfrac{7}{4}$

2. (a) $(x-2)^2 + (y+4)^2 = 25$ *or* $x^2 + y^2 - 4x + 8y - 5 = 0$
   (b) $(x+1)^2 + (y-3)^2 = 100$ *or* $x^2 + y^2 + 2x - 6y - 90 = 0$
   (c) $(x+1)^2 + (y-4)^2 = 85$ *or* $x^2 + y^2 + 2x - 8y - 68 = 0$

3. $(x+1)^2 + (y-4)^2 = 25$ *or* $x^2 + y^2 + 2x - 8y - 8 = 0$

4. Proof : *the sum of the radii equals the distance between the centres.*

5. Proof : *by substitution* , F(3,1)

6. Proof : *by substitution* , $y = -11x + 46$

7. (a) $2y = -x + 7$
   (b) (-1,4) and (3,2) , Proof.

8. (a) Proof : *by substitution*
   (b) $x + 8y = 4$
   (c) $x + 8y + 126 = 0$

The Straight Line (answers)

1. (a) $m = -\frac{1}{3}$ (b) $m = 4$

2. $a = -9$

3. $k = 2\frac{1}{2}$

4. (a) $y = 4x - 11$ (b) $x + 2y = 1$ (c) $3x + 2y = 2$

5. Proof : *gradients multiply to* -1

6. $y = 4x - 12$

7. (a) $x + 2y = 1$ (b) $12y = 11x + 50$

8. (a) $y = -3x - 1$ *and* $6y = 2x + 4$
   (b) $T(-\frac{1}{2}, \frac{1}{2})$
   (c) Proof

9. $x + 2y = 19$

Differentiation 1 (answers)

1. *(a)* $6x+6$ *(b)* $4x^3+2x^{-2}$ *(c)* $8x+12$

2. *(a)* $4x^3-5x^4$ *(b)* $6x-3x^{-4}$ *(c)* $-x^{-2}-\frac{2}{3}x^{-3}$

   (d) $\frac{5}{2}t^{3/2}+2t$ *(e)* $-\frac{4}{3}p^{-7/3}+\frac{1}{3}p^{-4/3}$ *(f)* $\frac{1}{2}u^{-1/2}-\frac{1}{4}u^{-3/2}$

   (g) $1-4x^{-3}$ *(h)* $\frac{5}{2}v^{-1/2}-v^{-3/2}$ *(i)* $\frac{1}{2}t^{-1/2}$

3. *(a)* 4 *(b)* -3

4. *(a)* 200 *(b)* 0 *(c)* -100 (pressure is decreasing)

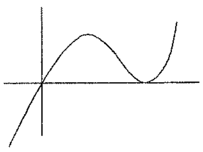
5. *(a)* $y=-8x-3$ *(b)* $6x+2y=8$

6. f is increasing when $0 < x < 1$.

7. (-2,32) , *Max.* , (0,0) , *Min.* , (1,5) , *Max.*

8. *(a)* $(\frac{2}{3}, \frac{32}{27})$, *Max.* , $(2, 0)$, *Min.*

   (b) $(0,0)$ and $(2,0)$

   (c) 

Differentiation 2 (answers)

1. *(a)* $12(3x+2)^3$ *(b)* $3x(x^2-4)^{1/2}$ *(c)* $-\dfrac{2}{(5-4x)^{1/2}}$

   (d) $-\dfrac{3(2x+3)}{(x^2+3x)^4}$ *(e)* $\dfrac{4}{(x-2)^2}$ *(f)* $-\dfrac{2x}{(x^2-7)^{3/2}}$

2. *(a)* $6\cos 3x$ *(b)* $-3\sin x\cos^2 x$ *(c)* $10\sin x\cos x$

   (d) $2\sin(3-2x)$ *(e)* $-\dfrac{\cos x}{\sin^2 x}$ *(f)* $\dfrac{\sin x}{2\cos^{3/2} x}$

3. $4x+5y=9$

4. *(a)* $y=\sqrt{3}$ *(b)* $\dfrac{dy}{d\theta} = \cos\frac{1}{2}\theta$, *value of* $\frac{1}{2}$ (c) Proof

5. *(a)* $100 - x$

   (b) 2500 square metres

6. *(a)* $h = \dfrac{4000}{x^2}$

   (b) Proof

   (c) 20 by 20 by 10 (since $x=20$ *and* $h=10$)

Quadratic Theory (answers)

1.

(a) $(x+3)^2-12$ (b) $(x-\frac{5}{2})^2-5\frac{1}{4}$ (c) $3(x+2)^2-10$

(d) $2(x-\frac{3}{2})^2-8\frac{1}{2}$ (e) $20-(x-4)^2$ or $-(x-4)^2+20$ (f) $3-2(x+1)^2$ or $-2(x+1)^2+3$

2. (a) $f(x)=2(x-4)^2-25$, $a=2,\ p=-4,\ q=-25$

min value of -25 @ $x=4$

(b) $g(x)=(\sin\alpha-\frac{1}{2})^2-\frac{5}{4}$, $p=-\frac{1}{2}$ *and* $q=-\frac{5}{4}$

min value of $-\frac{5}{4}$ @ $\alpha=\frac{\pi}{6}$

3. (a) $k=-1$

(b) i) $p=-4$ *or* $p=-2$ ii) $-4<p<-2$

(c) proof (discriminant a perfect square)

4. (a) Proof

(b) $k=-3$ *or* $k=2$

Systems of Equations (answers)

1. (a) { (-2,-5) , (5,2) } (b) { (6,5) , (10,3) }

(c) { (-13,-6) , (12,$\frac{13}{2}$) } (d) { $(-\frac{3}{5},-5\frac{4}{5})$, (1,-1) }

2. (a) { $(-\frac{3}{7},-\frac{4}{7})$, (3,2) } (b) { (-1,-1) , (1,4) }

3. (a) $x^2-4y-4=0$

(b) $x+2y=58$

(c) From the system $y=24$ *or* $y=35$, however $y=35$ must be discarded since this y value would make x -ve.
$\therefore$ $y=24$ *making* $x=10$ *sides are* 10 , 24 , 26.

4. $P(-\frac{7}{2},-\frac{11}{4})$ and $Q(3,7)$

Functions (answers)

1. (a) i) 17 ii) 12

(b) i) $9x^2 - 3x$ ii) $3a^2 + 3a - 1$ iii) $k^4 + 2k^3 + 2k^2 + k$

(c) $g^{-1}(x) = \dfrac{x+1}{3}$

2. (a) $f(h(x)) = \dfrac{2x^2}{3}$

(b) $f^{-1}(x) = \dfrac{2}{3x} + 1$, $x \neq 0$, $x \in R$

3. (a) $x \neq -\frac{1}{2}$, $x \in R$

(b) Proof

(c) $x = -1$ *or* $x = \frac{1}{2}$

4. (a) Proof

(b) Proof

Vectors (1) (answers)

1. (a) i) $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ ii) $\begin{pmatrix} -9 \\ 5 \end{pmatrix}$

(b) $\sqrt{40}$ *or* $2\sqrt{10}$

2. (a) i) $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ ii) $\begin{pmatrix} 16 \\ 4 \\ -16 \end{pmatrix}$

(b) $\sqrt{125}$ *or* $5\sqrt{5}$

3. $\sqrt{11}$

4. $S(-10,-5,3)$

5. Proof , $EF : FG = 1 : 2$

6. Proof (scalar product = 0)

7. magnitude of $\underset{\sim}{u} = 2$ $\therefore$ $\underset{\sim}{v} = \frac{1}{2}\underset{\sim}{u}$ $\rightarrow$ $\underset{\sim}{v} = \begin{pmatrix} \sqrt{2}/2 \\ -1/2 \\ 1/2 \end{pmatrix}$

8. scalar product $= 0$, vectors are perpendicular

Vectors (2) (answers)

1. (a) Proof (scalar product is -ve)

   (b) $\theta = 56 \cdot 7^{\circ}$

   (c) $k = 2$

2. $ABC = 24 \cdot 4^{\circ}$

3. (a) $p = 3$ and $q = 6$

   (b) $A(12, 0, 0)$, $C(0, 3, 0)$

   $\angle ACF = 29 \cdot 8^{\circ}$

Sequences & Recurrence Relations (answers)

1. (a) £373.85

   (b) 16 years (15·75)

2. (a) 28 , 26·4 , 25·12 , 24·096

   (b) because $-1 < a < 1$ (or equivalent)

   (c) Limit = 20

3. (a) $k = \frac{1}{2}$, $c = -20$

   (b) $U_4 = -33 \cdot 75$

4. (a) approx. 10 hours $(5 \times 2\,hours)$ .... this low value is reached before adding 12

   (b) No .... sequence has a limit of 24 , however this is an upper value i.e. lower limit is 12 (below 12·3)

5. (a) 60·64 units

   (b) After 3 hours the lower level climbs above 100 units .... this is o.k.

   After a number of calculations the limit formula can be applied and the limit found (this is an upper limit since b is involved).

   Limit = 254 units .... o.k. since well below 300 units.

   Conclusion : at any time antibiotic in bloodstream will be between 154 and 254 units. This is ideal.

Integration (answers)

1. (a) $3x^3 + 3x^2 + C$ (b) $3x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5 + C$ (c) $-2x^{-½} - x^{-1} + C$

2. (a) $21\frac{11}{15}$ (b) 8

3. C = -5 , $y = \frac{1}{2}x^2 + \dfrac{1}{x} - 5$

4. C = -4 , $f(x) = 6x + \dfrac{24}{x} - 4$

5. $4\frac{1}{2}$ *square units*

6. (a) 4 *square units*

   (b) Proof

7. (a) A(0,5) and B(3,8)

   (b) 9 *square units*

Polynomials (answers)

1. (a) $f(2) = 42$, $f(-1) = 0$ (b) $y = 3$

2. (a) $x^2 + 6x - 2$, *rem.* = 1

   (b) $x^2 + 3$, *rem.* = -4

   (c) $x^2 - x - 1$, *rem.* = 0

3. Proof , $(x-4)(x+1)$

4. Proof , $(x-3)(x-1)(x+3)$

5. (a) $(x-1)(x-4)(x+5)$ (b) $(x-1)(2x-3)(2x+1)$

6. (a) $k = -20$

   (b) $c = -2$

   (c) $a = -13$, $b = 6$

7. (a) $x = -⅔$, $x = 1$, $x = 2$

   (b) $x = -2$, $x = -1$, $x = 3$

8. Proof

9. (a) $V(h) = h^3 + 4h^2 + 3h$

   (b) Proof by synthetic division (or substitution) i.e. $V(5) = -ve$, $V(6) = +ve$

   (c) $h = 5{\cdot}5$

Trig. Equations (answers)

All answers in degrees except 1(b) and 4.

1. (a) $\{150^\circ , 330^\circ\}$ (b) $\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$

   (c) $\{105, 165, 285, 345\}$ (d) $\{23\cdot5, 96\cdot5, 143\cdot5\}$

2. (a) $\{19\cdot5, 160\cdot5, 270\}$ (b) $\{14\cdot5, 165\cdot5, 270\}$

   (c) $\{90, 270\}$ (d) $\{19\cdot5, 160\cdot5, 210, 330\}$

3. $\{7\cdot9, 115\cdot8\}$

4. $\{\pi/3, \pi\}$

5. $\{24\cdot2, 102\cdot7\}$

6. (a) $2\cos 2x + 2\sin 2x = 1 \quad \therefore \quad P = 2, Q = 2$ and $R = 1$

   (b) $\{57\cdot2\}$

7. (a) Proof

   (b) $\theta = 40^\circ$

The Exponential & Logarithmic Function (answers)

1. (a) $x = 1\cdot63$ (b) $x = 1\cdot78$

2. (a) 4 *ml*

   (b) 11 *min* 30 *sec*

   (c) 27 *min* 44 *sec*

3. $x = 16$

4. (a) $k = 0\cdot074$

   (b) 9 *hours* 22 *min*

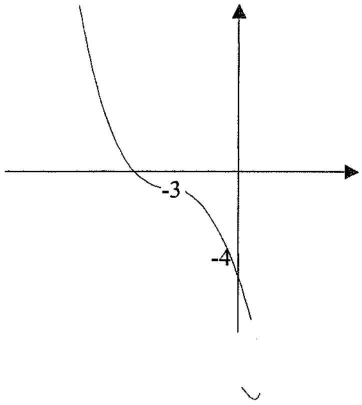
5. (a) Proof

   (b) $n = 3\cdot6$ and $k = 10\cdot4$ *or* $10\cdot8$

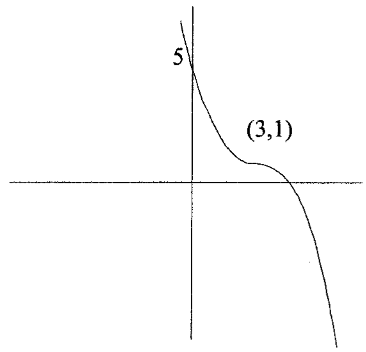
Graphicacy (answers)

1. (a) $y = 4\sin x^\circ$ (b) $y = 20\cos 2x^\circ$ (c) $y = 4\sin 4x^\circ + 2$

2. (a)



(b)



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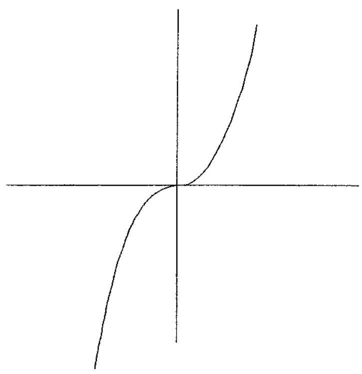

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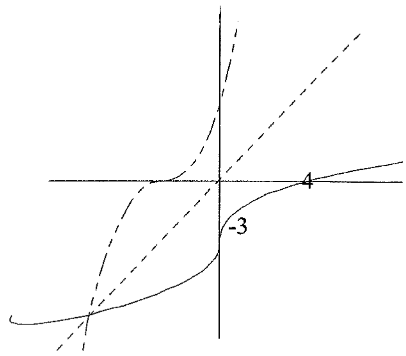
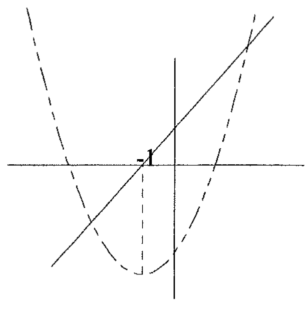
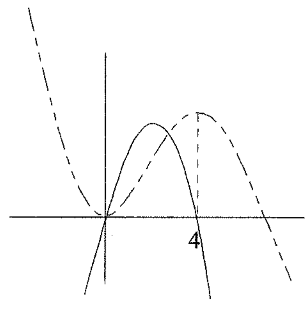


3.

(a)



(b)



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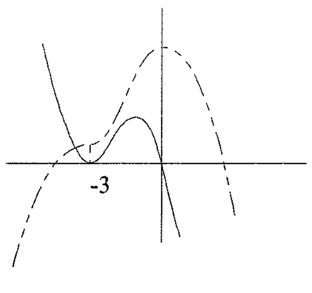

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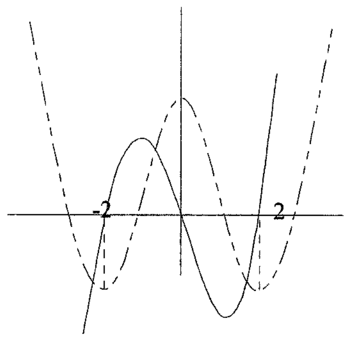


The Wave Function [$a\cos x + b\sin x$] (answers)

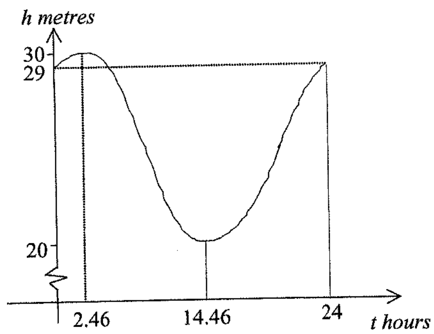
1. (a) $2\sin(x+60)^\circ$

   (b) min. of 2 @ $x = 30^\circ$

2. (a) $h = 5\cos(15t - 36\cdot9)^\circ + 25$

   (b) 20 metres @ 14 28

   (c)



   (d) 8 hours 51 minutes (from 2202 until 0653)

3. (a) Proof

   (b) $PR^2 = 12\cos\theta + 6\sin\theta + 14 \quad \therefore \quad a = 12,\ b = 6,\ c = 14$

   (c) $PR = 5\cdot2$, $\theta = 0\cdot46$ *radians*

4. (a) Proof

   (b) $\sqrt{13}\cos(x - 73\cdot9)^\circ$, $k = \sqrt{13}$ *and* $\alpha = 73\cdot9^\circ$

   (c) $\{\ 0^\circ\ ,\ 147\cdot8^\circ\ \}$

Mixed Exercise (1) (answers)

1. (a) Proof ; $(x+3)(x-2)(2x+1)$

   (b) $y = 4x + 4$

2. (a) $3x + y = 9$

   (b) (4,0)

3. (a) i) $\dfrac{dv}{dx} = \dfrac{1}{\sqrt{2x+1}}$ ii) $x = -\frac{12}{25}$

   (b) $f'(\frac{\pi}{4}) = -2$

4. (a) $x + 2y + 2 = 0$

   (b) Proof

   (c) T(-6,2)

5. (a) $P(k+4\ ,0\ ,k+2)$

   (b) $k = -3$

6. $k = -\frac{1}{3}$ or $k = \frac{1}{2}$

Mixed Exercise (2) (answers)

1. (a) Proof ; 1 : 2

   (b) $k = -\frac{2}{3}$

2. (a) Proof ; $(x-1)(x-3)(x+2)$

   (b) $12\frac{2}{3}$ *square units*

3. Proof

4. $y = 12x - 14$

5. $\{\, 48\cdot 6^\circ\ , 131\cdot 4^\circ\ , 270^\circ \,\}$

6. (a) At the end of day 4 (before catching fish)

   (b) No ...... min. 6kg since 10% of 6 = 0.6 (he eats what he catches)

Mixed Exercise (3) (answers)

1. (a) Proof (b) $k = \frac{1}{2}$ *or* $k = 1$

2. (a) $\frac{x^2}{2} + \frac{2}{x^{\frac{1}{2}}} + C$ (b) $\frac{dy}{dx} = 4x^3 + 2 - \frac{2}{x^3}$

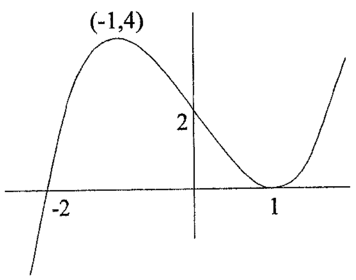
3. (a) Proof (b) Proof (c) Proof

   (d) $x = 1 \cdot 04$ and $w = 1 \cdot 39$ *metres*

4. $\angle ABC = 58 \cdot 8^\circ$

5. (a) Proof then $x = 1$ (b) (-1,4) , Max. , (1,0) , Min.

   (c)



Mixed Exercise (4) (answers)

1. (a) $2(x-3)^2 - 17$; $a = 2, b = -3, c = -17$

   (b) $9\frac{1}{4} - (x - \frac{5}{2})^2$; $p = 9\frac{1}{4}, q = -\frac{5}{2}$

2. $\cos\theta + \sqrt{3}\sin\theta = 2\cos(\theta - \frac{\pi}{3})$

3. (a) $y = \frac{100}{x^{\frac{2}{3}}}$ (b) $y = e^{2x}$

4. (a) $y = 2x - \frac{1}{10}x^2$ (b) *Volume* $= 4000$ *cubic metres* (shaded area $133\frac{1}{3}$ *square cm*)

5. (a) 11m , 5m

   (b) high tide (pm) 17 42 (17 43)

   (c) 12 hours 34 min. (4π *hours* $= 12 \cdot 56$ *hrs*)

6. (a)
$$4g + 4f + c = -8$$
$$4g + 2f + c = -5$$
$$6g - 2f + c = -10$$

   (b) $x^2 + y^2 - 11x - 3y + 20 = 0$

The Straight Line

1. Find the equation of the line which passes through the point P(3,-5) and is parallel to the line passing through the points (-1,4) and (7,-2).

2. Given that the points (3 , -2) , (4 , 5) and (-1 , a) are collinear , find the value of a .

3. Given that the lines with equations $x+4y=7$, $3x+y=10$ and $x-5y+a=0$ meet at the same point (i.e. they are concurrent), find the value of a .

4. On a coordinate diagram, a perpendicular is drawn from the origin to the line with equation $x+2y=15$.
Find the coordinates of the point of contact.

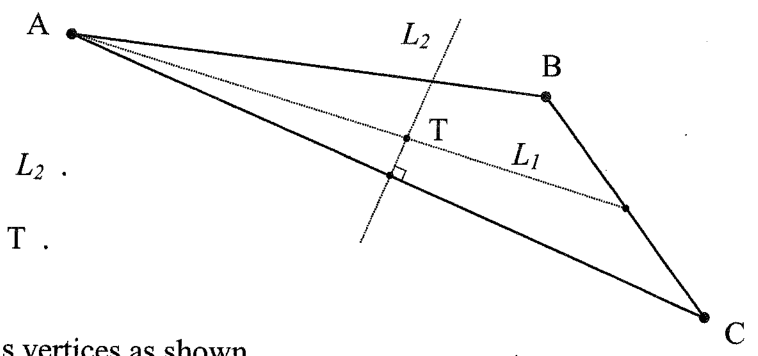
5. PQRS is a rhombus where vertices P , Q and S have coordinates (-5 , -4) , (-2 , 3) and (2 , -1) respectively.

Establish the coordinates of the fourth vertex R , and hence, or otherwise, find the equation of the diagonal PR.

6. Find the equation of the line which passes through the point (a,1) and is perpendicular to the line with equation $x + \frac{y}{a} = 1$, and state where this perpendicular crosses the axes.

7. Triangle ABC has as its vertices A(-18,6) , B(2,4) and C(10,-8) .

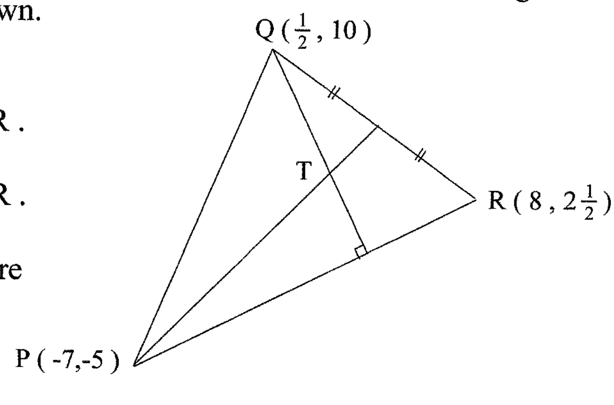
L_1 is the median from A to BC. L_2 is the perpendicular bisector of side AC.



(a) Find the equations of L_1 and L_2 .

(b) Hence find the coordinates of T .

8. In the diagram below triangle PQR has vertices as shown.



(a) Find the equation of the median from P to QR .

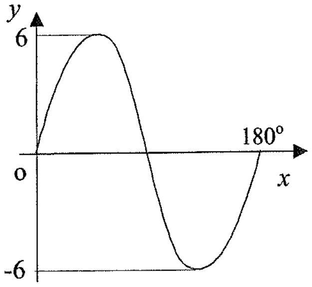
(b) Find the equation of the altitude from Q to PR .

(c) Hence find the coordinates of the point T where these two lines cross.

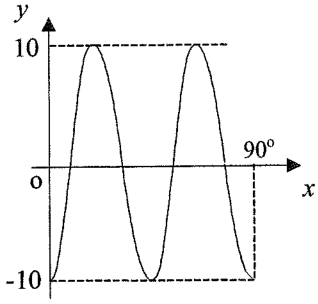
Graphicacy

1. Write down an equation which describes the relationship between x and y in each graph below :

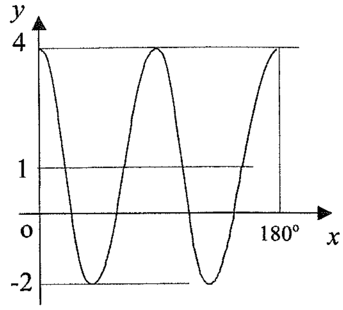
(a)



(b)



(c)



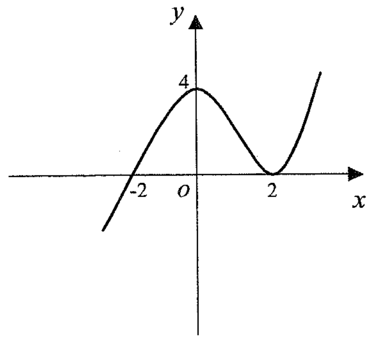
2. The graph of $y = f(x)$ is shown opposite.

(a) Draw a sketch of $y = -f(x)$.

(b) Draw a sketch of $y = f(-x)$.

(c) Draw a sketch of $y = f(x+2)$.

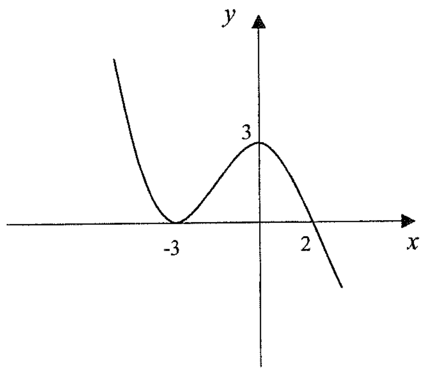
(d) Draw a sketch of $y = 4 - f(x)$.



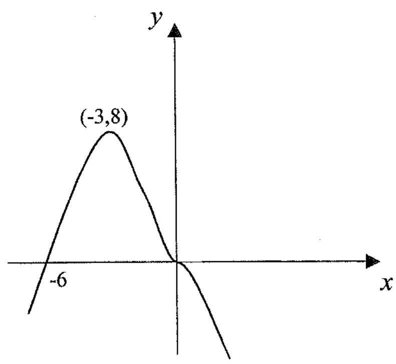
3. For each graph below $y = f(x)$

Draw a sketch of the graph of the derivative, $y = f'(x)$, of each function.

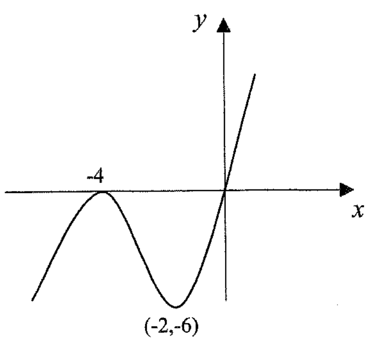
(a)



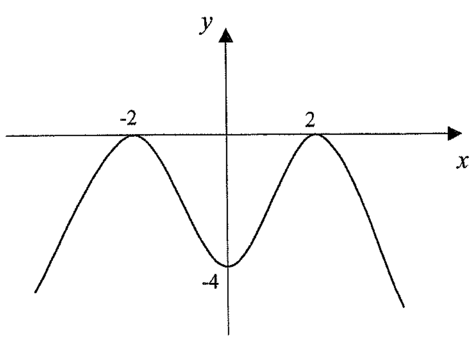
(b)



(c)



(d)



Differentiation 2

1. Given that $g(x) = \left(\frac{1}{2}x^2 - 1\right)^4$, find the value of $g'(2)$.

2. Find the derivative , with respect to x , of $\frac{1}{6}(1+2x)^3 + \sin 2x$.

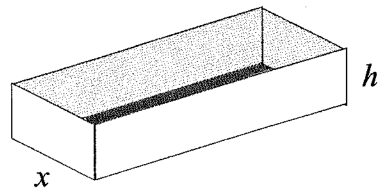
3. A function is defined as $f(\theta) = \sin^2\theta - \cos 2\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$.

   (a) Show that $f'(\theta) = 3\sin 2\theta$.

   (b) Hence calculate the rate of change of the function at $\theta = \dfrac{\pi}{12}$.

4. A tank has to be designed for the top of a water tower. It must be rectangular in shape and open at the top.
Its base sides are to be in the ratio 1 : 3.

   (a) Taking the shorter of the base sides to be x , show that the tanks surface area (A) can be expressed as



$$A = 3x^2 + 8xh ,$$

   where h is the height of the tank.

   (b) Given that the tank has to constructed from 144 sq. metres of steel plate construct a formula for h in terms of x .

   (c) Hence, or otherwise, show that the volume of the tank, in terms of x , is given as

$$V(x) = 54x - \tfrac{9}{8}x^3 .$$

   (d) Find the dimensions of the tank which will maximise its volume.

5. A function is defined as $g(\theta) = 2\sin 2\theta - 4\cos^2\theta$.

   Show that $g'(\theta)$ can be written in the form

$$g'(\theta) = 4(\cos 2\theta + \sin 2\theta)$$

6. Given $f(x) = \dfrac{1}{1 - 2x}$ where $x \ne \frac{1}{2}$, find the value of $f'(-1)$.

Functions

1. Two functions f and g are defined on the set of real numbers as follows :

$$f(x) = 2x - 3 \quad , \quad g(x) = \frac{x + 9}{4} \quad .$$

(a) Evaluate $f\big(g(-3)\big)$.

(b) Find an expression , in its simplest form, for $g\big(f(x)\big)$.

(c) Hence verify that $f^{-1}(x) = g\big(f(x)\big)$

2. A function is given as $g(x) = \dfrac{2}{\sqrt{x}}$.

(a) State a suitable domain for this function on the set of real numbers.

(b) Find a formula for $g^{-1}(x)$ the inverse function of $g(x)$.

(c) Hence, find, in its simplest form, a formula for $f(x)$,
given that $f(x) = g^{-1}\big(g^{-1}(x)\big)$.

3. The functions $f(x) = x^2 + 3$ and $h(x) = 7 + 3x$ are defined on the set of real numbers.

(a) Evaluate $h\big(f(2)\big)$.

(b) Find an expression , in its simplest form, for $f\big(h(x)\big)$.

(c) For what values of x would the functions f and h produce the same image ?

4. A function in terms of x is given as

$$f(x) = 3x(x - 1) + (3a + 3) \text{ , where } a \text{ is a constant.}$$

Given that $k = a + 1$ show that $f(k) = 3k^2$.

5. Two functions are defined as $f(x) = px^2 - 1$ and $h(x) = \dfrac{5x + q}{2}$,
where p and q are constants.

(a) Given that $f(2) = h(2) = 7$, find the values of p and q .

(b) Find $h\big(f(x)\big)$.

(c) Find the value of the constant k when $2\big[h\big(f(x)\big)\big] - 4 = k\big[f(x)\big]$.

Trig. Equations and Compound Angle Formulae

1. Solve the equation $\sqrt{6}\sin 2\theta - \sqrt{3} = 0$ for $0 \le \theta < 2\pi$

2. Solve the equation $2\sqrt{2}\sin(2\theta + \frac{\pi}{6}) - \sqrt{6} = 0$, for $0 \le \theta < \frac{\pi}{2}$

3. Solve algebraically the equation

$$\cos 2x^\circ - 6\sin x^\circ + 7 = 0 \quad \text{for} \quad 0 \le x < 360.$$

4. Solve algebraically the equation

$$3\cos 2x^\circ - \cos x^\circ - 2 = 0, \quad 0 \le x < 360.$$

5. If $\tan\alpha = \frac{1}{\sqrt{5}}$, where $\pi \le \alpha < \frac{3\pi}{2}$, find the **exact** value of $\sin 2\alpha$.

6. (a) Given that $\tan\theta = \frac{\sqrt{5}}{\sqrt{9}}$, where $0 < \theta < \frac{\pi}{2}$, find the **exact** value(s) of $\sin\theta$ and $\cos\theta$.

(b) Hence show that the exact value of $\sin 2\theta = \frac{3}{7}\sqrt{5}$.

7. Triangle ABC is right-angled at B .

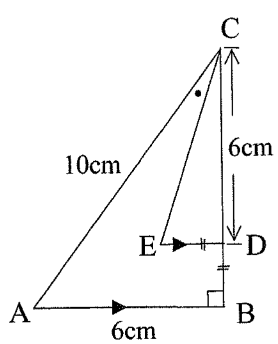
Given that $\sin A = \frac{2\sqrt{ab}}{a+b}$, show that the perimeter (P) of the triangle can be expressed as $P = 2\left(a + \sqrt{ab}\right)$.

8. The diagram opposite is a sketch of the tail - fin of a model plane.

ED is parallel to AB.
ED = DB.

Show that the **exact** value of Cos ACE is given as

$$\text{Cos ACE} = \frac{3\sqrt{10}}{10}.$$



Integration (1)

1. Evaluate $\int_0^2 (4-3x)^2\,dx$

2. Given that $\int_1^a (5-2x)\,dx = 2$.

   Find **algebraically** the two possible values of a .

3. A curve has as its derivative $\frac{dy}{dx} = 2 - 12x$.

   Given that the point *(1,3)* lies on this curve, express y in terms of x.

4. A certain curve has as its derivative $\frac{dy}{dx} = \frac{(x-2)(x+2)}{x^2}$.

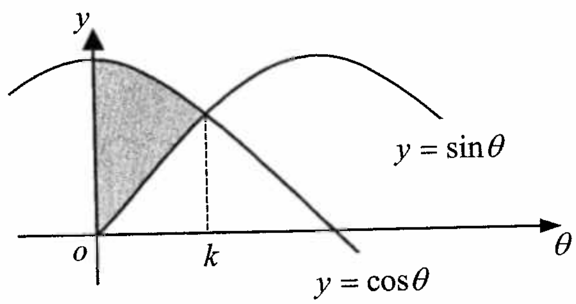
   (a) Given that $y=2$ when $x = 4$, find a formula for y in terms of x .

   (b) Hence, or otherwise, find a second value of x which makes $y = 2$.

5. Evaluate $\int_{-1}^{1} \left(2 + \frac{1}{x^2}\right)^2 dx$

6. Given that $\frac{dy}{dx} = 3x - \frac{1}{x^3}$, and if $y = 0$ when $x = 1$, find y when $x = 2$.

7. The sketch below shows part of the graphs of $y = \sin\theta$ and $y = \cos\theta$.



   (*a*) Write down the value of k in radians.

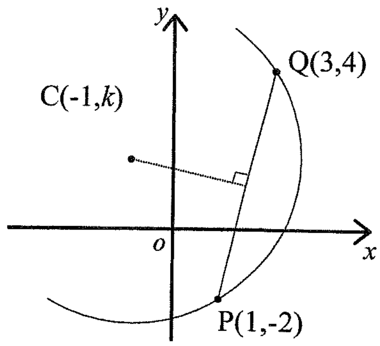
   (*b*) Hence show that the exact area of the shaded region is $\sqrt{2} - 1$ square units.

The Circle

1. A circle has as its equation $x^2 + y^2 - 4x + 3y + 5 = 0$.

   Find the equation of the tangent at the point (3,-2) on the circle.

2. A circle has as its centre C(-1,k) .
   A chord PQ is drawn with end - points P(1,-2) and Q(3,4) as shown in the diagram.



   (a) Find the value of k .

   (b) Hence, establish the equation of the circle .

3. Two circles, which do not touch or overlap, have as their equations

$$(x-15)^2 + (y-6)^2 = 40 \quad \text{and} \quad x^2 + y^2 - 6x - 4y + 3 = 0.$$

   (a) Show that the **exact** distance between the centres of the two circles is $4\sqrt{10}$ units.

   (b) Hence show that the shortest distance between the two circles is equal to the radius of the smaller circle.

4. A clothes manufacturer has asked an advertising company to design a small logo which can be stamped onto their garments to prove authenticity. The design company has come up with the following simple logo which consists of a kite within a circle (diagram 1). A schematic of the logo produces diagram 2.

diagram 1

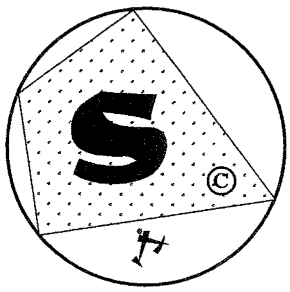
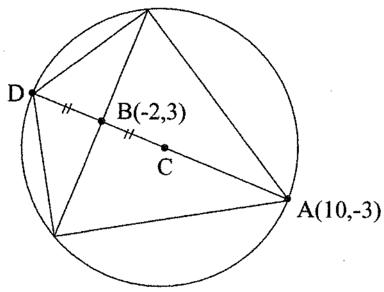


diagram 2

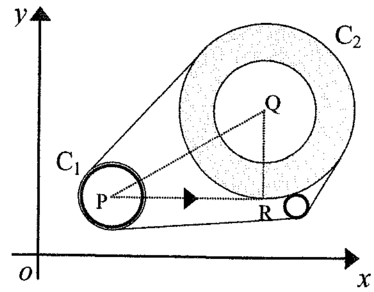


   Relative to a set of rectangular axes the points A and B have coordinates (10,-3) and (-2,3) as shown and CB = BD.

   (a) Find the coordinates of point C, the centre of the circle.

   (b) Hence establish the equation of the circle.

5. The diagram opposite represents part of a belt driven pulley system for a compact disc turntable .



   P and Q are the centres of the circles C_1 and C_2 respectively.
   PR is parallel to the x-axis and is a tangent to circle C_2. **PQ = 6 units** .
   The coordinates of P are $(2\cdot8\ , 2)$.

   (a) Given that angle QPR = 30° , establish the coordinates of the point Q , rounding any calculations to one d. p. where necessary.

   (b) Hence write down the equation of circle C_2 .

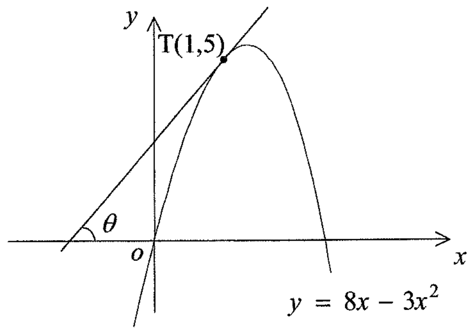
Differentiation 1

1. Differentiate $\dfrac{x^2 + 1}{\sqrt{x}}$, with respect to x ,

expressing your answer with positive indices.

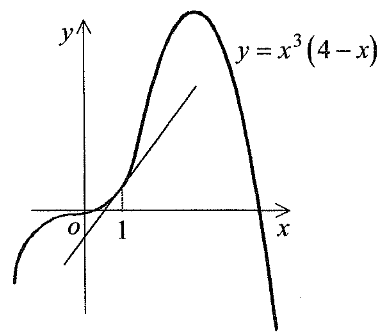
2. Find $f'(x)$ when $f(x) = \dfrac{x^3 - 6\sqrt{x}}{x^2}$.

3. The diagram below shows the parabola with equation $y = 8x - 3x^2$ and the line which is a tangent to the curve at the point T(1,5).



Find the size of the angle marked θ , to the nearest degree.

4. The sketch shows part of the graph of $y = x^3(4-x)$.
The tangent at the point where $x = 1$ is also shown.



(a) Find the equation of the tangent.

(b) Given that the axes are not to scale, calculate the true anti-clockwise angle between this tangent and the x - axis, giving your answer correct to the nearest degree.

5. Show that the function $f(x) = 4(1-2x)^3$ is decreasing for **all** values of x , except $x = \frac{1}{2}$.

6. Show that the curves with equations $y = x^2 + 8x + 3$ and $y = 1 + 4x - x^2$ touch each other at a single point, and find the equation of the common tangent at this point.

7. A function is given as $f(x) = 5x^5 + 10x^3 + 9x$ and is defined on the set of real numbers.

(a) Show that the derivative of this function can be expressed in the form $f'(x) = (ax^2 + b)^2$ and write down the values of a and b .

(b) Hence explain why this function has no stationary points and is in fact increasing for **all** values of x .

Quadratic Theory

1. An equation is given as $tx + 3t + \dfrac{t+5}{x} = 0$ where $x \neq 0$ and $t \neq 0$.

   (a) Show clearly that this equation can be written in the form

   $$tx^2 + 3tx + t + 5 = 0 \, .$$

   (b) Hence find the values of t which would result in the above equation having **real roots**.

2. For what values of t does the equation $x^2 + (t-1)x = 2t + 1$ have no real roots ?

3. A function is defined as $f(x) = \dfrac{3p}{x^2 - 18x + 87}$, for $x \in R$ and p is a constant.

   (a) Express the function in the form $f(x) = \dfrac{3p}{(x-a)^2 + b}$, and hence state the maximum value of f in terms of p.

   (b) Given now that $p = \dfrac{2}{\sqrt{2}+1}$ show that the **exact** maximum value of f is $\sqrt{2} - 1$.

4. A function is given as $f(x) = \sin^2\theta + 2\sin\theta - 3$ for $0 \leq \theta \leq 2\pi$.

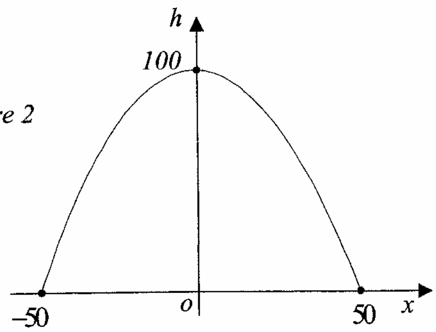
   (a) Express the function in the form $f(x) = (\sin\theta + a)^2 + b$ and write down the values of a and b.

   (b) Hence, or otherwise, state the minimum value of this function and the corresponding replacement for θ.

5. The famous Gateway Arch in the United States is parabolic in shape. *Figure 2* shows a rough sketch of the arch relative to a set of rectangular axes.

Figure 1

Figure 2



   From *figure 2* establish the equation connecting h and x.

6. What can you say about b if the equation $x + \dfrac{4}{x} = b$ has **real** roots ?

Vectors (1)

1. (a) Points E, F and G have coordinates (-1,2,1) , (1,3,0) and (-2,-2,2) respectively. Given that $3EF = GH$, find the coordinates of the point H.

   (b) Hence calculate $\left|\overrightarrow{EH}\right|$.

2. Two vectors are defined as $F_1 = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $F_2 = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$.

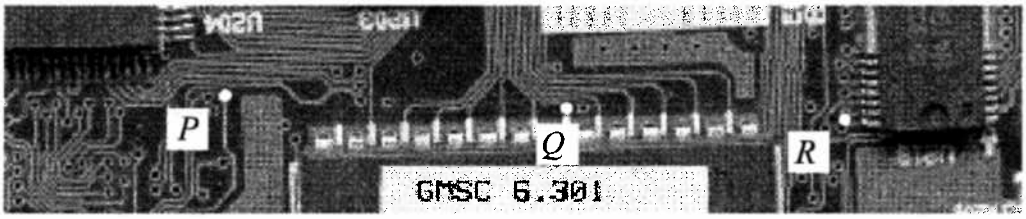
   Show that these two vectors are perpendicular.

3. Find a ***unit*** vector parallel to the vector $V = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

4. A and B are the points (-2,-1,4) and (3,4,-1) respectively.

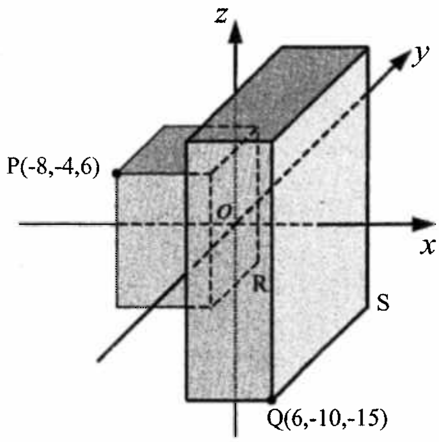
   Find the coordinates of the point C given that $\dfrac{AC}{CB} = \dfrac{3}{2}$.

5. The picture below shows a small section of a larger circuit board.



   Relative to rectangular axes the points P , Q and R have as their coordinates (-8,3,1) , (-2,-6,4) and (2,-12,6) respectively.
   Prove that the points P, Q and R are collinear, and find the ratio $PQ : QR$.

6. A metal casting is in the shape of two cuboids.
   The casting is positioned relative to a set of rectangular axes as shown below. Both cuboids have been centred along the x-axis.



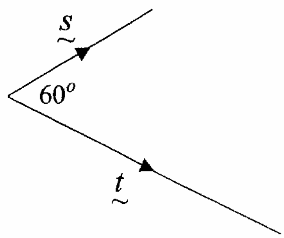
   (*a*) Given that corners P and Q have coordinates $(-8,-4,6)$ and $(6,-10,-15)$ respectively .
   Write down the coordinates of corners R and S.

   (*b*) Hence show that the corners P , R and S are collinear.

   (*c*) Find $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.

   (*d*) Calculate the size of $\angle$PRQ .

7. Consider the diagram opposite . The magnitudes of vectors $\underset{\sim}{s}$ and $\underset{\sim}{t}$, are 1 unit and 2 units respectively. The angle between the two vectors is 60^o as shown.

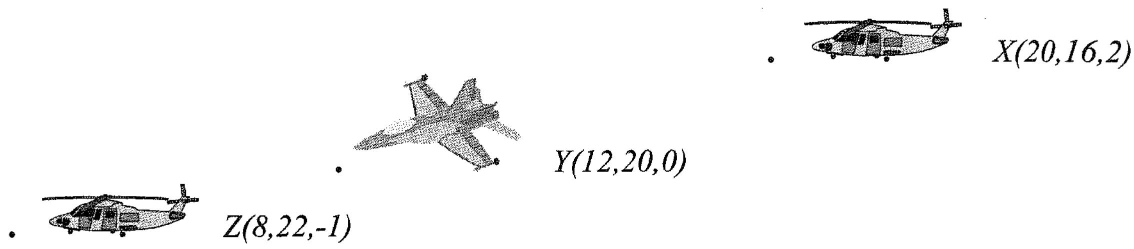


   (a) Given that $\underset{\sim}{u} = 2\underset{\sim}{s}$ and $\underset{\sim}{v} = (\underset{\sim}{t} - \underset{\sim}{s})$.
   Evaluate the scalar product $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

   (b) What can you say about the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$?

Vectors (2)

1. Three military aircraft are on a joint training mission. Their positions relative to each other, within a three dimensional framework, are shown in the diagram below :



(a) Show that the three aircraft are collinear .

(b) Given that the actual distance between Z and Y is 42km , how far away from Z is X ?

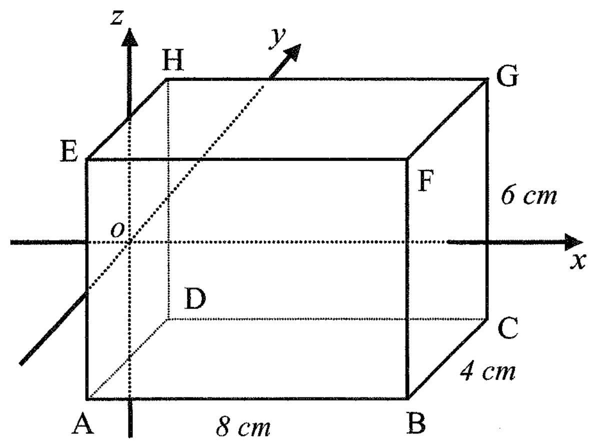
(c) Following further instructions aircraft Y moves to a new position (50,15,-8) . The other two aircraft remain where they are . For this new situation , calculate the size of $\angle XYZ$.

2. A cuboid is placed relative to a set of coordinate axes as shown in the diagram. The cuboid has dimensions *8 cm* by *4 cm* by *6 cm.*

The origin of the axes is at the intersection point of the diagonals ED and HA.
1 unit represents *1 cm* .

(a) Find the coordinates of B , D and G.

(b) Hence calculate the size of angle BDG.



3. Three vertices of the quadrilateral $PQRS$ are $P(7,-1,5)$, $Q(5,-7,2)$ and $R(-1,-4,0)$.

(*a*) Given that $\overrightarrow{QP} = \overrightarrow{RS}$, establish the coordinates of S.

(*b*) Hence show that SQ is perpendicular to PR .

4. Two vectors are defined as $V_1 = 4\underset{\sim}{i} + \underset{\sim}{j} + \sqrt{8}\,\underset{\sim}{k}$ and $V_2 = 8\,\underset{\sim}{i} + \sqrt{24}\,\underset{\sim}{j} + a\sqrt{3}\,\underset{\sim}{k}$,

where a is a constant and <u>all</u> coefficients of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are greater than zero..

(*a*) Given that $|V_2| = 2|V_1|$, calculate the value of a.

(*b*) Hence prove that the angle between the vectors V_1 and V_2 is acute .

Sequences and Recurrence Relations

1. A sequence of numbers is defined by the recurrence relation $U_{n+1} = kU_n + c$, where k and c are constants.

   (a) Given that $U_0 = 10$, $U_1 = 14$ and $U_2 = 17 \cdot 2$, find **algebraically** , the values of k and c .

   (b) Calculate the value of E given that $E = \frac{3}{4}(L - 2)$, where L is the limit of this sequence.

2. A sequence is defined by the recurrence relation $U_{n+1} = 0 \cdot 8\, U_n + 3$.

   (a) Explain why this sequence has a limit as $n \to \infty$.

   (b) Find the limit of this sequence.

   (c) Taking $U_0 = 10$ and L as the limit of the sequence, find n such that

$$L - U_n = 2 \cdot 56$$

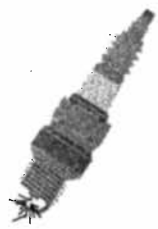
3. A sequence is defined by the recurrence relation $U_{n+1} = aU_n + b$, where a and b are constants.

   (*a*) Given that $U_0 = 4$ and $b = -8$, express U_2 in terms of a .

   (*b*) Hence find the value of a when $U_2 = 88$ and $a > 0$.

   (c) Given that $S_3 = U_1 + U_2 + U_3$, calculate the value of S_3 .

4. Over a period of time the effectiveness of a standard spark plug slowly decreases. It has been found that, in general, a spark plug will loose 8% of its burn efficiency every **two months** while in average use.



   (a) A new spark plug is allocated a *Burn Efficiency Rating (BER)* of 120 units.
   What would the *BER* be for this plug after a year of average use ?

   Give your answer correct to one decimal place.

   (b) After exhaustive research, a new fuel additive was developed.
   This additive , when used at the end of every **four month** period, immediately allows the *BER* to increase by 8 units.

   A plug which falls below a *BER* of 80 units should immediately be replaced.

   What <u>should</u> be the maximum recommended lifespan for a plug , in months, when using this additive ?

Integration (2)

1. Integrate each of the following functions with respect to x :

$a)$ $3 - \frac{1}{4}x$ $b)$ $x^2 - 1 - \dfrac{2}{x^2}$ $c)$ $\dfrac{2 - x^3}{x^2}$

$d)$ $\sqrt{x} - \dfrac{1}{\sqrt{x}}$ $e)$ $\dfrac{x^2 - x - 1}{x^{\frac{2}{3}}}$ $f)$ $(x + 1)^3$

2. For a certain curve, $\dfrac{dy}{dx} = x^2 - 3x$.

(a) Find y in terms of x if the point $(3, -\frac{5}{2})$ lies on the curve.

(b) Hence find y when $x = -1$.

3. For a certain curve, $\dfrac{dy}{dx} = x^{\frac{2}{3}} + \dfrac{1}{x^{\frac{2}{3}}}$.

(a) Given that the point $(1, \frac{3}{5})$ lies on this curve, find y in terms of x.

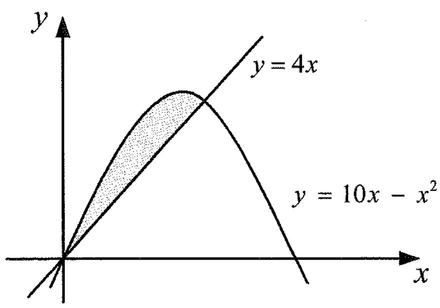
(b) Hence find the value of y when $x = 8$.

4. Evaluate :

$$\int_0^3 (x - 3)^2 \, dx$$

5. In the diagram opposite the curve has as its equation $y = 10x - x^2$ and the line $y = 4x$.

Calculate the shaded area in square units.



6. Calculate the area enclosed between the curves $y = 4x - x^2$ and $y = x^2 - 4x + 6$.

Polynomials

1. Given that $x-1$ is a factor of x^3+kx^2-5x+6 , find the value of k and hence fully factorise the expression.

2. Given that $x = -1$ and $x = 2$ are two roots of the equation $x^3 + ax^2 + 2x + b = 0$, establish the values of a and b and hence find the third root of the equation.

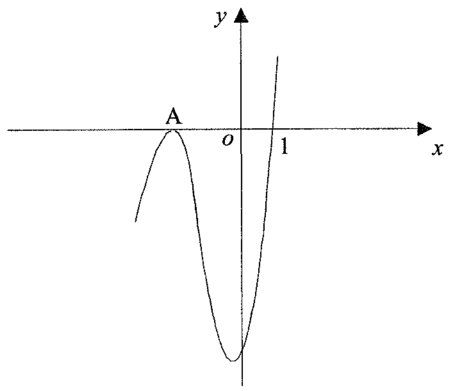
3. Show that $x+2y$ is a factor of $x^2+(2y+3)x+6y$.

4. Solve the equation $x^3 - 4x^2 + x + 6 = 0$.

5. Find the value of c if $x+1$ is a factor of the expression

$$cx^3 + (1-c)x^2 - (2c+1)x + c$$

6. The curve shown below has as its equation $y=x^3+5x^2+kx-9$.



(a) Given that the curve crosses the x - *axis* at the point $(1,0)$, find the value of k .

(b) Hence find the coordinates of the point A

7. When $x^3 - bx^2 + 3x - 2$ and $3x^3 + 2x^2 - b^2x - 4$ are both divided by $x-2$ the remainders are equal.
Find the two possible values of b .

8. Show that $x+2a$ is a factor of $x^3-7a^2x-6a^3$ and find the other factors.

The Exponential & Logarithmic Function

1. a) Given that $2\log_x y = \log_x 2y + 2$ find a relationship connecting x and y.

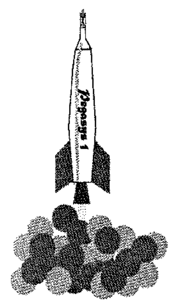
   (b) Hence find y when $x = \frac{1}{4}y$.

2. Given that $\log_2(x+1) - \log_2 x = \log_2 8$, find the value of x .

3. By taking logarithms in the base two of both sides of the following equation, find algebraically the value of x

$$8^{x+1} = 4^{2x-3}$$

4. A rockets fuel payload decays according to the formula



$$M_t = M_0 e^{-0 \cdot 04t} ,$$

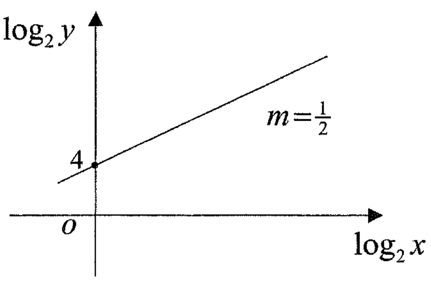
where M_0 is the initial payload in tonnes and M_t is the fuel remaining after t seconds.

Calculate, to the nearest second, how long the rocket takes to use up 80% of its fuel load.

5. A radioactive substance decays according to the formula $M_t = M_o.8^{-0 \cdot 3t}$, where M_o is the intitial mass of the substance, M_t is the mass remaining after t years.

Calculate , to the nearest day, how long a sample would take to **half** its original mass.

6. The diagram, which is not drawn to scale, shows part of a graph of $\log_2 x$ against $\log_2 y$.
The straight line has a gradient of $\frac{1}{2}$ and passes through the point (0,4).



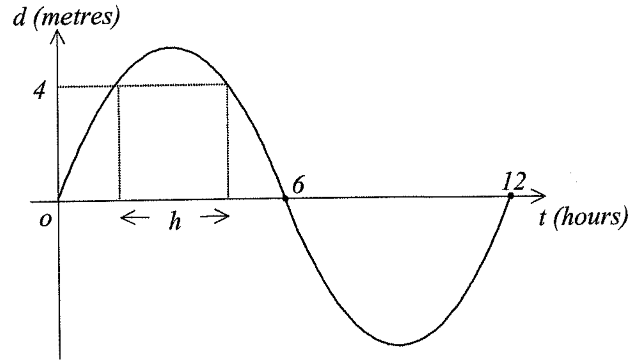
Express y in terms of x .

The Wave Function ($a\cos x + b\sin x$)

1. (a) Express $12\cos x^\circ + 5\sin x^\circ$ in the form $k\sin(x-\alpha)^\circ$, where k and α are constants and $k > 0$.

   (b) Hence state the **minimum** value of h given that $h(x) = \dfrac{8}{12\cos x^\circ + 5\sin x^\circ + 3}$.

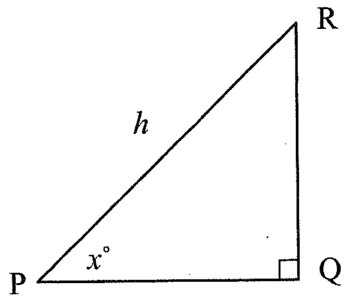
2. The diagram below shows the depth of water in a small harbour during a standard 12-hour cycle.



It has been found that the depth of water follows the relationship $d = 6\sin 30t$, where d is the depth, in metres, above or below a mean height and t is the time elapsed, in hours, from the start of the cycle.

Calculate the value of h , the length of time, in hours and minutes, that the depth in the harbour is greater than or equal to 4 metres.
Give your answer to the nearest minute.

3. Triangle PQR is right-angled at Q. Angle QPR $= x^\circ$ and side PR $= h$ units as shown.



   (a) Express RQ and PQ in terms of h and x° .

   (b) Given that the perimeter of the triangle is $19\cdot24$ units and $h = 8$ units, show that
   $$2\sin x^\circ + 2\cos x^\circ = 2\cdot81$$

   (c) Express $2\sin x^\circ + 2\cos x^\circ$ in the form $k\cos(x-\alpha)^\circ$ where $k > 0$ and $0 < \alpha < 90$.

   (d) Hence solve the equation $2\sin x^\circ + 2\cos x^\circ = 2\cdot81$ for x where $x > 40$.

4. The luminosity, L units, emitting from a pulsing light source is given by the formula

   $L = \cos 36t^\circ + \sqrt{3}\sin 36t^\circ + 2$, where t is the time in seconds from switch on.

   (a) Express L in the form $R\cos(36t-\alpha)^\circ + 2$, where $R > 0$ and $0 \le \alpha \le 360$.

   (b) Sketch the graph of L for $0 \le t \le 10$, indicating all relevant points.

   (c) If $L = 2\cdot5$ the light source has the same luminosity as the surrounding light. When will this **first** occur during this ten second period ?

No Topic (Algebra & Trigonometry)

1. (a) Express $\dfrac{1}{\sqrt{a}+2} + \dfrac{1}{\sqrt{a}-2}$ in the form $\dfrac{k\sqrt{a}}{a+l}$

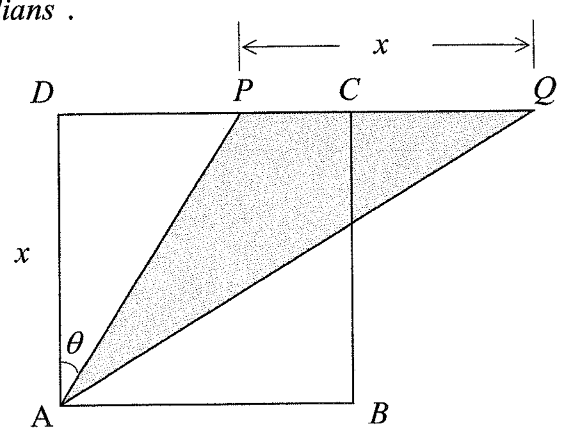
and write down the values of k and l.

(b) Hence solve the equation $\dfrac{1}{\sqrt{a}+2} + \dfrac{1}{\sqrt{a}-2} = \dfrac{6}{5}$

for a, where a is **not** a integer.

2. If $x = \dfrac{a+y}{b-1} = \dfrac{a-y}{b+1}$, find x in terms of a and b.

3. In the diagram below $ABCD$ is a square of side x cm . P and Q are points on DC and DC produced such that PQ is equal to x cm.
$\angle PAD = \theta$ *radians* .



(a) Express the length of AP in terms of x and θ .

(b) Show that angle APQ is $\frac{\pi}{2} + \theta$ *radians* .

(c) Given that $\sin(\frac{\pi}{2} + \theta) = \cos\theta$, show that the ratio of the area of $\Delta\ APQ$ to the area of the square $ABCD$ is $1 : 2$.

4. Given that $\sin\theta = \dfrac{\sqrt{2}}{3\sqrt{3}}$ *where* $0 < \theta < \frac{\pi}{2}$,

show that the exact value of $\cos\theta$ is $\frac{5}{9}\sqrt{3}$.

5. Make x the subject of the formula $\dfrac{1+ax}{1+bx} = \dfrac{c}{d}$.

Simplify your result if $c = b$ and $d = a$.

Mixed Exercise (Unit 1)

1. The perpendicular bisector of the line joining the points A(1,-2) and B(3,-4) passes through the point $(5,k)$.
Find the value of k.

2. A line passes through the points $(\frac{1}{2}, \frac{a}{2})$ and (b, ab).

   (*a*) Show that the gradient of this line is a.

   (*b*) Hence show that no matter what values a or b take, this line will always pass through the origin.

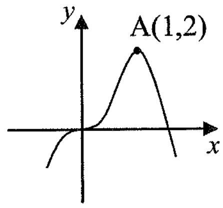
3. Two functions are defined on suitable domains and are given as

$$f(x) = x + 3 \qquad \textit{and} \qquad g(x) = x^2 - 1.$$

   (*a*) Find an expression for the function h when $h(x) = g(f(x))$.

   (*b*) Find the values of a given that $h(a) = f(a) + 1$

4. The diagram shows the graph of $y = g(x)$. Point A has coordinates (1,2)



Sketch the graph of $y = -g(x) + 2$, indicating clearly the new position of the point A.

5. A function is defined as $h(d) = \left[\dfrac{-4}{d^2 - 4d + 5}\right] + 6$, for $0 \le d \le 5$.

   (a) Express this function in the form $h(d) = \left[\dfrac{-4}{(d-a)^2 + b}\right] + 6$.

   (b) Hence state the minimum value of h and the corresponding value of d.

6. Differentiate $\dfrac{1}{2x}(4\sqrt{x} + 1)$ with respect to x.

7. Show that the function $f(x) = x(x-3)^2 + 3x$ is **never** decreasing.
For what value of x is the function stationary ?

8. From the moment a magnetic source is brought in contact with a second magnet, it dissipates at the rate of 7% every 30 minutes.

   (a) If its initial strength is 200 *mfu* (magnetic flux units), what strength will be remaining after 2 hrs?

   (b) At the end of each 2 hour period, the magnet is passed through a a very intense electric field. This allows the magnet to regain some of its lost strength. It gains 16 *mfu* per pass, every 2 hrs. If this is continued, how long will it take for the magnet to first drop below a strength of 106 *mfu*?

   (c) The magnet cannot fulfill its function if it drops below a strength of 60 *mfu*. Will this situation ever occur for this magnet over the long term?
   Your answers should be accompanied by the appropriate working and justifications.

Mixed Exercise (Unit 2)

1. Solve the equation $x^3 + 4x^2 + x - 6 = 0$.

2. Find the value of k if $x-2$ is a factor of the expression

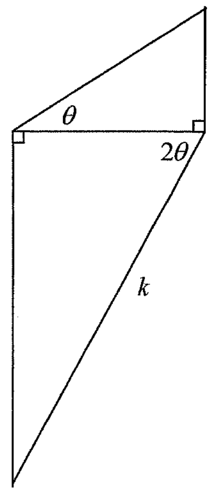
$$x^3 + (k+1)x^2 - kx - 18 \ .$$

3. Find the value(s) of t if the equation $x + \dfrac{16}{x} = t + 2$ has **equal** roots ?

4. Evaluate $\displaystyle\int_{-1}^{1} x + \frac{1}{x^2} \; dx$

5. Solve the equation $2\cos 2\theta + 3\sin\theta = \sin\theta$ for $0 \le \theta < 2\pi$.

6. Consider the diagram opposite where both θ and 2θ are acute.



(*a*) Given that $\tan\theta = \frac{1}{\sqrt{2}}$, find the **exact** value of

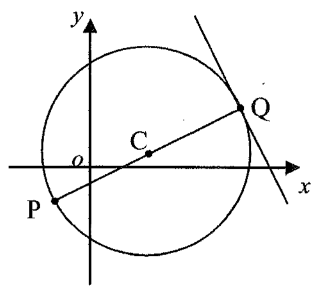
i) $\cos\theta$

ii) $\cos 2\theta$.

(*b*) Hence find the **exact** value of k .

7. Find the equation of the circle whose diameter joins the points A(-2,5) and B(4,1).

8. A circle has as its centre C(5,1) with the point P(-4,-2) on its circumference as shown in the diagram below.



(*a*) Establish the coordinates of the point Q , given that PQ is a diameter of the circle.

(*b*) Hence find the equation of the tangent to the circle at the point Q.

9. (*a*) Show that the point P(1,3) lies on the circumference of the circle with equation $x^2 + y^2 + 4x - 2y - 8 = 0$.

(*b*) The point Q(0,k) also lies on the circumference of this circle. Find the value of k where $k > 0$.

(*c*) A second circle passes through the points P , Q and R(-2,0). Find the equation of this second circle.

Mixed Exercise (Unit 3)

1. Triangle ABC has as its vertices A(2,-1,0) , B(1,-2,1) and C(3,0,k).

   Given that $\angle ABC$ is a right-angle, find the value of k.

2. Find the unit vector , $\underset{\sim}{u}$, which is parallel to the vector $\underset{\sim}{a} = \begin{pmatrix} -3 \\ \sqrt{2} \\ \sqrt{5} \end{pmatrix}$.

3. Find :

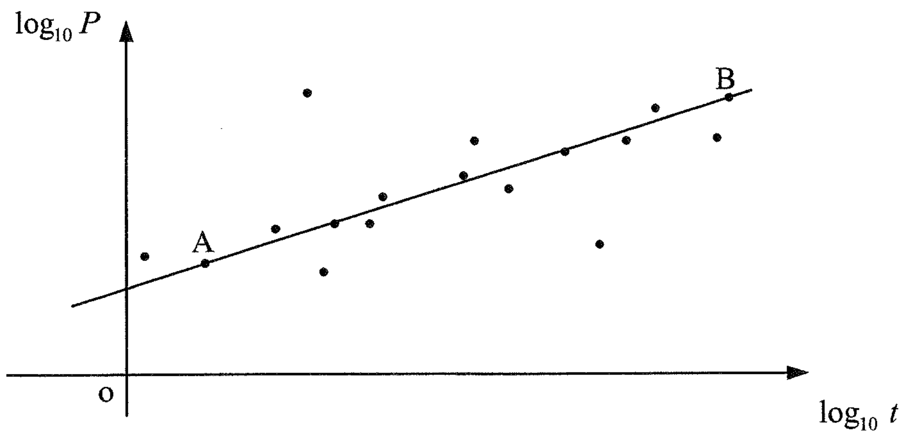
   (a) $\int (3x-1)^4 \,.\, dx$ (b) $\int \frac{1}{\sqrt{(1+2x)}} \,.\, dx$

   (c) $\int \sin 4\theta \,.\, d\theta$ (d) $\int 2 + \cos 2\theta \,.\, d\theta$

4. A curve has as its equation $y = \dfrac{1}{x-4}$.

   Find the equation of the tangent to this curve at the point where $x = 3$.

5. In a scientific experiment, corresponding readings of pressure (P) and temperature (t) were taken and recorded.
   The data was put into logarithmic form and the graph of $\log_{10} P$ against $\log_{10} t$ was plotted.
   This resulting graph, with the line of best fit added, can be seen below.



   (a) Explain why the original data takes the form $P = kt^n$, where k and n are constant integers.

   (b) Given that the points A and B on the graph have coordinates $(1\cdot6 , 11\cdot98)$ and $(1\cdot844 , 13\cdot45)$ respectively, establish the equation of the line and state the values of n and k .

6. (a) Express $2\sin x^\circ + 2\cos x^\circ$ in the form $k\cos(x-\alpha)^\circ$ where $k > 0$ and $0 < \alpha < 90$.

   (b) Hence solve the equation $2\sin x^\circ + 2\cos x^\circ = 2\cdot81$ for x where $x > 40$. Give your answer correct to the nearest degree.

The Circle (answers)

1. $y = 2x - 8$

2. (a) $k = 2$ (b) $(x-2)^2 + (y+1)^2 = 20$

3. (a) Pyth. or dis. form. to answer - $4\sqrt{10}$ (b) d $= \sqrt{10}$ = the radius of the smaller circle

4. (a) C(2,1) (b) $(x-2)^2 + (y-1)^2 = 80$ or $x^2 + y^2 - 4x - 2y - 75 = 0$

5. (a) Q(8,5) (b) $(x-8)^2 + (y-5)^2 = 9$

The Straight Line (answers)

1. $3x + 4y + 11 = 0$

2. $a = -30$

3. $a = 2$

4. (3 , 6)

5. R(5 , 6) , $y = x + 1$

6. $y = \dfrac{1}{a}x$, the origin.

7. (a) $L_1 \Rightarrow y = -\frac{1}{3}x$; $L_2 \Rightarrow y = 2x + 7$ (b) T(-3,1)

8. (a) $y = x + 2$ (b) $y + 2x = 11$ (c) T(3,5)

Differentiation 1 (answers)

1. $\frac{3}{2}x^{\frac{1}{2}} - \dfrac{1}{2x^{\frac{3}{2}}}$

2. $f'(x) = 1 + 9x^{-\frac{5}{2}}$

3. $\theta = 63^\circ$

4. (a) $y = 8x - 5$ (b) $\theta = 83^\circ$

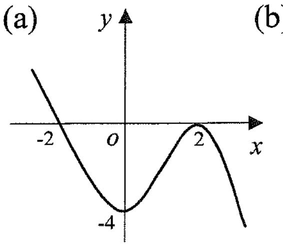
5. proof

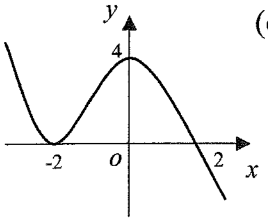
6. T(-1,-4) , $y = 6x + 2$

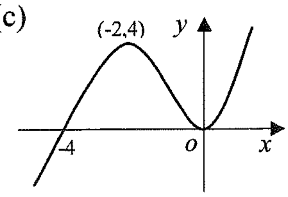
7. (a) a = 5 and b = 3 (b) Explanation $f'(x)$ must be > 0 since a perfect square $\Rightarrow$ function is never decreasing

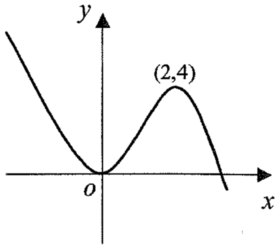
Graphicacy (answers)

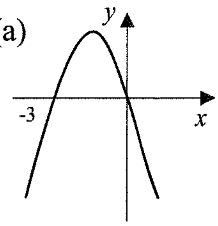
1. (a) $y = 6\sin 2x$ (b) $y = -10\cos 8x$ (c) $y = 3\cos 4x + 1$

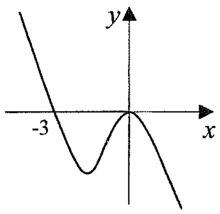
2. (a)


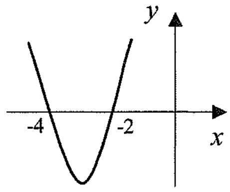
(b)


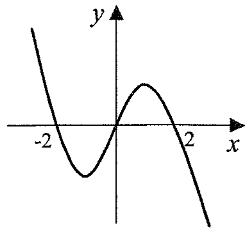
(c)


(d)


3. (a)


(b)


(c)


(d)


Differentiation 2 (answers)

1. $g'(2) = 8$

2. $\frac{1}{2}(1+2x)^2 \, . \, 2 \quad + \quad 2\cos 2x$

3. (a) proof (b) rate of change = $\frac{3}{2}$ (or equiv.)

4. (a) proof (b) $h = \dfrac{144 - 3x^2}{8x}$ (c) proof (d) $4cm$ *by* $12cm$ *by* $3cm$

5. proof

6. $f'(-1) = \frac{2}{9}$

Quadratic Theory (answers)

1. (a) proof (b) $t \leq 0$ or $t \geq 4$

2. $-5 < t < -1$

3. (a) $f(x) = \dfrac{3p}{(x-9)^2 + 6}$, $\therefore$ $f_{\max} = \dfrac{3p}{6} = \frac{1}{2}p$ (b) $f_{\max} == \dfrac{\sqrt{2}-1}{1} = ans.$

4. (a) $(\sin\theta + 1)^2 - 4$, $\therefore$ $a = 1$ *and* $b = -4$ (b) $\theta = \dfrac{3\pi}{2}$

5. $h = \frac{1}{25}\left(2500 - x^2\right)$

6. b cannot lie between -4 *and* 4

Functions (answers)

1. (a) $f\left(g(-3)\right) = 0$ (b) $g\left(f(x)\right) = \frac{1}{2}(x + 3)$ (c) For $f^{-1}(x) = \frac{1}{2}(x + 3) = g\left(f(x)\right)$

2. (a) $x > 0$ (b) $g^{-1}(x) = \dfrac{4}{x^2}$ (c) $g^{-1}\left(g^{-1}(x)\right) = \dfrac{4}{\left(\frac{4}{x^2}\right)^2} = \dfrac{x^4}{4}$

3. (a) $h\left(f(2)\right) = 28$ (b) $f\left(h(x)\right) = 9x^2 + 42x + 52$ (c) $x = -1$ or $x = 4$

4. proof

5. (a) $p = 2$ and $q = 4$ (b) $h\left(f(x)\right) = \frac{1}{2}(10x^2 - 1)$ (c) $k = 5$

Vectors (1) (answers)

1. (a) $H(4,1,-1)$ (b) $\sqrt{30}$

2. proof; $F_1 \cdot F_2 = 0$

3. $\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ *or* $\frac{1}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

4. C(1 , 2 , 1)

5. proof ; $PQ : QR = 3 : 2$

6. (a) R(0,4,-6) , S(6,10,-15) (b) proof (c) $\overrightarrow{RP} = \begin{pmatrix} -8 \\ -8 \\ 12 \end{pmatrix}$ *and* $\overrightarrow{RQ} = \begin{pmatrix} 6 \\ -14 \\ -9 \end{pmatrix}$ (d) $\angle$ PRQ = $98 \cdot 7^\circ$

7. (a) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ (b) They are perpendicular

Vectors (2) (answers)

1. (a) Proof (b) 126 km (c) $\angle XYZ = 11 \cdot 7^\circ$

2. (a) B(8,-2,-3) , D(0,2,-3) and G(8,2,3) (b) $\angle$BDG = $44 \cdot 3^\circ$

3. (a) S(1, 2, 3) (b) proof

4. (a) $100 = 88 + 3a^2 \therefore 3a^2 = 12 \Rightarrow a = 2$ (b) "since scalar product >0, then angle acute"

Sequences & Recurrence Relations (answers)

1. (a) $k = 0 \cdot 8$ *and* $c = 6$ (b) $E = 21$

2. (a) Explanation i.e. $-1 < a < 1$ (b) $L = 15$ (c) $n = 3$

3. (a) $U_2 = a(4a-8) - 8 = 4a^2 - 8a - 8$ (b) $\therefore\ a = 6$ (c) $S_3 = 16 + 88 + 520 = 624$

4. (a) $72 \cdot 8\,units$ (b) 16 months .. (pupils must know to look at low value, i.e.... before adding 8) (4×4)

Integration (1) (answers)

1. 8

2. $a = 2$ or $a = 3$

3. $y = 2x - 6x^2 + 7$

4. (a) $y = x + \dfrac{4}{x} - 3$ (b) $x = 1$

5. $\frac{1}{3} + (-\frac{1}{3}) = -\frac{2}{3}$

6. $y = 4\frac{1}{8}$

7. (a) $k = \frac{\pi}{4}$ (b) proof ; $A = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$ *sq.units*

Integration (2) (answers)

1. $a)\ 3x - \dfrac{x^2}{8} + C \quad b)\ \dfrac{x^3}{3} - x + \dfrac{2}{x} + C \quad c)\ -\dfrac{x^2}{2} - \dfrac{2}{x} + C \quad d)\ \dfrac{2x^{\frac{3}{2}}}{3} - 2x^{\frac{1}{2}} + C$

   $e)\ \frac{3}{7}x^{\frac{7}{3}} - \frac{3}{4}x^{\frac{4}{3}} - 3x^{\frac{1}{3}} + C \quad f)\ \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

2. (a) $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2$ (b) $y = \frac{1}{6}$

3. (a) $y = \frac{3}{5}x^{\frac{5}{3}} + 3x^{\frac{1}{3}} - 3$ (b) $y = 22\frac{1}{5}$

4. 9

5. 36 square units.

6. $2\frac{2}{3}$ square units.

Polynomials (answers)

1. $(x - 1)(x - 3)(x + 2)$
2. $a = -5$ and $b = 8$; third root .... $x = 4$
3. proof
4. $x = -1$, $x = 2$, $x = 3$
5. $c + 2 = 0 \therefore c = -2$
6. (a) $k = 3$ (b) A(-3,0)
7. $b = -2$ or $b = 4$
8. proof : other factors are $x + a$ and $x - 3a$

Trig Equations (answers)

1. $\theta = \pi/8 , 3\pi/8 , 9\pi/8 , 11\pi/8$
2. $\theta = \frac{\pi}{12}$ *or* $\theta = \frac{\pi}{4}$
3. $\sin x^\circ = -4$ *or* $\sin x^\circ = 1 \therefore \sin x^\circ = -4$ has no solution , $x = 90^\circ$
4. $x = 0 , 146 \cdot 4 , 213 \cdot 6$
5. $\sin 2\alpha = \frac{1}{3}\sqrt{5}$
6. (a) $\sin\theta = \frac{\sqrt{5}}{\sqrt{14}}$ *and* $\cos\theta = \frac{\sqrt{9}}{\sqrt{14}}$ (b) proof
7. $P = (a + b) + (a - b) + (2\sqrt{ab}) = 2(a + \sqrt{ab})$
8. proof

Exponential & Logarithms (answers)

1. (a) $x^2 = \frac{1}{2}y \Rightarrow y = 2x^2$ (b) $y = \frac{1}{8}y^2 \Rightarrow$ ans. $y = 8$
2. $x = \frac{1}{7}$
3. $x = 9$
4. 40 seconds
5. $t = 406$ days
6. $y = 16x^{\frac{1}{2}}$ or $16\sqrt{x}$

The Wave Function (answers)

1. (a) $12\cos x^\circ + 5\sin x^\circ = 13\sin(x - 292\cdot6)$ (b) $h_{\min} = \dfrac{8}{(13) + 3} = \frac{1}{2}$

2. h = 3 hours 12 min

3. (a) $RQ = h\sin x^\circ$ and $PQ = h\cos x^\circ$ (b) $8 + 8\sin x + 8\cos x = 19\cdot24 \therefore 2\sin x + 2\cos x = 2\cdot81$
(c) $2\sin x^\circ + 2\cos x^\circ = \sqrt{8}\cos(x-45)^\circ$ (d) $x = 52^\circ$ (*note:* $x - 45 = 353$ *produces an ans* $< 40^\circ$)

4. (a) $L = 2\cos(36t - 60)^\circ + 2$ (b) sketch (c) $t = 3\cdot8$ seconds

No - Topic (answers)

1. (a) $\dfrac{2\sqrt{a}}{a-4}$, $k = 2$ *and* $l = -4$ (b) $a = 16$ ($a = 1$ is discarded)

2. $x = \dfrac{a}{b}$

3. (a) $AP = \dfrac{x}{\cos\theta}$ (b) proof (c) $A_\Delta : A_{sq} = \frac{1}{2}x^2 : x^2 = 1 : 2$

4. proof

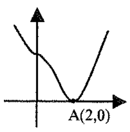
5. $x = \dfrac{c - d}{ad - bc}$; $x = -\dfrac{1}{a + b}$

Mixed Exercise - Unit 1 (answers)

1. $k = 0$

2. (a) proof... $m = \dfrac{2ab - a}{2b - 1} = \dfrac{a(2b-1)}{2b-1} = a$ (b) proof to $y = ax$ $\therefore$ always through origin

3. (a) $h(x) = x^2 + 6x + 8$ (b) $a = -4$ *or* $a = -1$

4. 

5. (a) $h(d) = \left[\dfrac{-4}{(d-2)^2 + 1}\right] + 6$ (b) $h_{\min} = 2$ @ $d = 2$.

6. $\dfrac{d}{dx} = -x^{-3/2} - \frac{1}{2}x^{-2}$

7. $f'(x) = 3x^2 - 12x + 12$ $\therefore$ a perfect square and always increasing. Stat. @ $x = 2$.

8. (a) 149.61 *mfu*

   (b) Pupils must know to watch the <u>low</u> value i.e. before b is added. ans: at the end of 6 hours.

   (c) Again the limit is a high value $\therefore$ Limit − 16 = 47.51, which is below 60 .... the magnet will become unusable in the long term.

Mixed Exercise - Unit 2 (answers)

1. $x=-3$, $x=-2$, $x=1$

2. $k=3$

3. $t=-10$ or $t=6$.

4. -2

5. $\{ \frac{\pi}{2} , \frac{7\pi}{6} , \frac{11\pi}{6} \}$

6. (a) i) $\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$ ii) $\cos 2\theta = \frac{1}{3}$ (b) $k = 3\sqrt{2}$

7. $(x-1)^2+(y-3)^2=13$ *or* $x^2+y^2-2x-6y-3=0$

8. (a) Q(14,4) (b) $3x+y=46$

9. (a) proof (b) $k=4$ (c) $(x+1)^2+(y-2)^2=5$

Mixed Exercise - Unit 3 (answers)

1. $k=5$

2. $\therefore \quad \underset{\sim}{u} = \frac{1}{4}\begin{pmatrix} -3 \\ \sqrt{2} \\ \sqrt{5} \end{pmatrix}$

3. (a) $\frac{1}{15}(3x-1)^5 + C$ (b) $(1+2x)^{\frac{1}{2}} + C$ (c) $-\frac{1}{4}\cos 4\theta + C$ (d) $2\theta + \frac{1}{2}\sin 2\theta + C$

4. $y = -x + 2$

5. (a) proof (b) $n=6$, $k = 240$

6. (a) $\sqrt{8}\cos(x-45)^\circ$ (b) $x = 52^\circ$, (*note:* $x-45 = 353$ *produces an ans* $< 40^o$)