GAUSSIAN ELIMINATION

Simultaneous Equations

We already know how to solve two equations in two unknowns.

Now we are going to solve equations in three unknowns. To be able to do this we need three equations. We can then solve simultaneously by eliminating one letter at a time.

Example

Solve the system of equations $x + 2y - z = 7$ ... ①

$3x - y + 4z = -7$ ... ②

$2x + 3y + 2z = 4$ ... ③

Eliminate z

① × 4 $\quad 4x + 8y - 4z = 28$

② $\quad 3x - y + 4z = -7$

add $\quad 7x + 7y = 21$

① × 2 $\quad 2x + 4y - 2z = 14$

③ $\quad 2x + 3y + 2z = 4$

$\quad 6x + 7y = 18$

Solve $\quad 7x + 7y = 21$ ... ④

$\quad 6x + 7y = 18$ ... ⑤

Subtract $\quad 3x = 3$

$x = 1$

In ④ $\quad 7 + 7y = 21$

$y = 2$

In ① $\quad 1 + 4 - z = 7$

$z = -2$

Solution $x = 1,\ y = 2,\ z = -2$

To make this method more efficient we can put the equations into matrix form.

Example

Using the equations from the previous example

$$x + 2y - z = 7$$

$$3x - y + 4z = -7$$

$$2x + 3y + 2z = 4$$

In matrix form this gives

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 3 & -1 & 4 & -7 \\ 2 & 3 & 2 & 4 \end{array}\right)$$

This is called the **AUGMENTED MATRIX**

We now perform elementary row operations

- a row can be multiplied by a constant
- rows can be added or subtracted
- rows can be interchanged

We are aiming to get the matrix in the following form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

$* =$ number

get these to equal zero

This is called **UPPER TRIANGULAR FORM**

We then revert back to equations and solve by back substitution.

This method of solving the equations is called **GAUSSIAN ELIMINATION**

So returning to the equations

$$x + 2y - z = 7$$

$$3x - y + 4z = -7$$

$$2x + 3y + 2z = 4$$

We solve as follows

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 3 & -1 & 4 & -7 \\ 2 & 3 & 2 & 4 \end{array}\right)$$

$$\begin{array}{l} \\ R2 - 3R1 \\ R3 - 2R1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -7 & 7 & -28 \\ 0 & -1 & 4 & -42 \end{array}\right)$$

← leave row 1 the same.

} use row 1 to get 0 at start of rows 2 and 3. (without using fractions!!)

Show what you are doing

$$\begin{array}{l} \\ \\ 7R3 - R2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -7 & 7 & -28 \\ 0 & 0 & 21 & -42 \end{array}\right)$$

← use row 2 to get 0, 0 at start of row 3.

Change back to equations.

$$21z = -42$$
$$z = -2$$

R2
$$-7y + 7z = -28$$
$$-7y - 14 = -28$$
$$-7y = -14$$
$$y = 2$$

R1
$$x + 2y - z = 7$$
$$x + 4 + 2 = 7$$
$$x = 1$$

Solution $x = 1,\ y = 2,\ z = -2$

Example

Solve using Gaussian elimination

$$2y + 3z = 4$$
$$x + y + z = 2$$
$$4x + 2y + 3z = 6$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 3 & 6 \end{array}\right)$$

Since first row starts with 0 swap with either R2 or R3.

$R3 \leftrightarrow R1$
$$\left(\begin{array}{ccc|c} 4 & 2 & 3 & 6 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \end{array}\right)$$

$R1 - 4R2$
$$\left(\begin{array}{ccc|c} 4 & 2 & 3 & 6 \\ 0 & -2 & -1 & -2 \\ 0 & 2 & 3 & 4 \end{array}\right)$$

$R2 + R3$
$$\left(\begin{array}{ccc|c} 4 & 2 & 3 & 6 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

R3 gives
$$2z = 2$$
$$z = 1$$

R2 gives
$$-2y - z = -2$$
$$-2y - 1 = -2$$
$$-2y = -1$$
$$y = \frac{1}{2}$$

R1 gives
$$4x + 2y + 3z = 6$$

$$4x + 1 + 3 = 6$$

$$4x = 4$$

$$x = \frac{1}{2}$$

Solution $x = \frac{1}{2}, \; y = \frac{1}{2}, \; z = 1$

Tips for Success….

- State what you are doing in each stage at the side of the matrix.
- Get in the form $\left(\begin{array}{ccc|c} - & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{array}\right)$ first using R1 with R2 and R3
- Get in form $\left(\begin{array}{ccc|c} - & - & - & - \\ 0 & - & - & - \\ 0 & 0 & - & - \end{array}\right)$ next using R2 with R3.
- Be careful with negative signs when subtracting
- Avoid fractions.

Maths in Action Book p265 Ex14.4 Q1(a)(c)(e),2,3,4

Types of Solutions

All the examples we have looked at so far for solving 3 equations in 3 variables have **one unique solution** (i.e we get one answer for each of the letters)

This is not always the case and three possibilities occur :- **one unique solution**
inconsistent equations
redundant equation

(a) One unique solution

- occurs when we obtain one value for each variable, as seen so far
- Geometrical meaning – 3 planes meeting at a point. (see vectors unit 3)

(b) Inconsistent equations

- occur when a row in the matrix looks like

$$\left.\begin{matrix} 0 & 0 & 0 \end{matrix}\ \right|\ k$$

leading to e.g. :-

- Geometrical meaning – 3 planes not all meeting together eg 2 parallel planes, planes form prism (see vectors unit 3)

Inconsistent equations give **NO SOLUTION**

(c) Redundant equation

- occur when a row in the matrix looks like :-

$$\left.\begin{matrix} 0 & 0 & 0 \end{matrix}\ \right|\ 0$$

- Geometrical meaning – actually only two distinct equations meeting in a line.

A redundant equation leads to an INFINITE NUMBER OF SOLUTIONS

Examples

Solve by Gaussian elimination

$$x + 2y + 2z = 11$$
$$2x - y + z = 8$$
$$3x + y + 3z = 18$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right)$$

$$\begin{array}{l} \\ R2 - 2R1 \\ R3 - 3R1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & -5 & -3 & -15 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R3 - R2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

not possible. since $0z \neq -1$

so this set of equations is inconsistent. There are <u>no</u> solutions.

<u>Example</u>

Solve by Gaussian Elimination $x + 2y + 2z = 11$
$x - y + 3z = 8$
$4x - y + 11z = 35$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{array}\right)$$

$$\begin{array}{l} \\ R2-R1 \\ R3-4R1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & 3 & -9 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R3-3R2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is a <u>redundant</u> row.
There are infinitely many solutions → use equations to find x and y in terms of z

R2 $-3y + z = -3$
$$y = \frac{z+3}{3}$$

R1 $x + 2y + 2z = 11$
$$x + \frac{2}{3}(z+3) + 2z = 11$$
$$3x + 2z + 6 + 6z = 33$$
$$3x = -8z + 27$$
$$x = \frac{27-8z}{3}$$

Solution $x = \dfrac{27-8z}{3}$, $y = \dfrac{z+3}{3}$ $z = z$

Exam Question Example

Use Gaussian elimination to reduce the system of equations

$$\begin{aligned} 2x - y + \alpha z &= 1 \\ x - y + 2z &= -3 \\ -x + 2y - 3z &= 2 \end{aligned}$$

to upper triangular form.

Explain what happens when $\alpha = 3$.

Find the solution corresponding to $\alpha = -13$.

$$\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1 \\ 1 & -1 & 2 & -3 \\ -1 & 2 & -3 & 2 \end{array}\right)$$

$$\begin{array}{l} \\ 2R2-R1 \\ 2R3+R1 \end{array} \left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1 \\ 0 & -1 & 4-\alpha & -7 \\ 0 & 3 & -6+\alpha & 5 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R3+3R2. \end{array} \left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1 \\ 0 & -1 & 4-\alpha & -7 \\ 0 & 0 & 6-2\alpha & -16 \end{array}\right)$$

When $\alpha = 3$ we have.

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1 & 1 & -7 \\ 0 & 0 & 0 & -16 \end{array}\right)$$

inconsistent raw $\Rightarrow$ no solutions.

When $\alpha = -13$ we have.

$$\left(\begin{array}{ccc|c} 2 & -1 & -13 & 1 \\ 0 & -1 & 17 & -7 \\ 0 & 0 & 32 & -16 \end{array}\right)$$

so $32z = -16$

$z = \frac{-1}{2}$.

R2 $-y + 17z = -7$

$-y = -7 + \frac{17}{2}$

$y = -\frac{3}{2}$.

R1 $2x - y - 13z = 1$

$2x + \frac{3}{2} + \frac{13}{2} = 1$

$2x = -7$

$x = \frac{-7}{2}$

Solution $x = \frac{-7}{2}$, $y = -\frac{3}{2}$ $z = -\frac{1}{2}$

Tips for success .....

- For equations containing an unknown letter (eg α) as a coefficient, solve first in terms of the letter ie find x, y, z in terms of α.
- If then given a number for the letter can substitute to get a particular solution.
- redundant row $0\ 0\ 0 \mid 0$ ⇒ infinite number of solutions. (write x, y in terms of z)
- inconsistent row $0\ 0\ 0 \mid \text{number}$ ⇒ no solutions.

Exercise :- MiA AH p268 Ex 14.6 Q1(a)(b)(c), 2, 3, 4

The Stability of Gaussian Elimination

For a method to be stable, if a small error is introduced (eg rounding) there must only be a small change in the solution.

Consider $100x + 99y = 199$ --- ①
$99x + 98y = 197$ --- ②

① × 99 $9900x + 9801y = 19701$
$9900x + 9800y = 19700$
Subtract $y = 1$
In ① $100x + 99 = 199$
$x = 1$

Now changing the original equations the original equations slightly

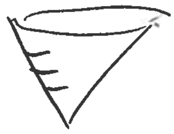
$100x + 99y = 200$ --- ③ ← increased by 1.
$99x + 98y = 197$ --- ④

③ × 99 $9900x + 9801y = 19800$
④ × 100 $9900x + 9800y = 19700$
subtract $y = 100$
In ③ $100x + 9900 = 200$
$100x = -9700$
$x = -97$.

A small change has drastically changed the solution. We say these equations are **<u>ill-conditioned.</u>**

A stable problem is well conditioned.

An unstable problem is ill conditioned.



small changes
$\rightarrow$ unstable .

Mia AH p273/274 Ex 14.9 Q 3, 4 + 2012 Exam Paper Question 14

MATRICES

Definition/Notations/Conventions

- A matrix is a rectangular array or arrangement of numbers.
- Each number in the array is called an **element** of the matrix.
- Matrices are arranged in rows and columns, and their **order** is given as **number of rows $\times$ number of columns.**

e.g. $\begin{pmatrix} 2 & -1 \\ 0 & 6 \\ -5 & 3 \end{pmatrix}$ This matrix has 3 rows and 2 columns. It is a matrix of **order 3×2.**

In general, an $m \times n$ matrix has m rows and n columns.

- The position of an element in a matrix can be described using the notation a_{ij}. This describes an element, a, of matrix A which is in the i'th row and j'th column.

e.g. in the matrix A= $\begin{pmatrix} 1 & 5 & 1 \\ 2 & 3 & 0 \\ 4 & -2 & -3 \end{pmatrix}$, a_{23}=0, the element of the 2nd row, 3rd column.

- A **column matrix** has only one column, e.g. $\begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix}$

- A **row matrix** has only one row, e.g. $\begin{pmatrix} 0 & -5 & 2 & 1 \end{pmatrix}$

- A **square matrix** has equal numbers of rows and columns, e.g. a 2×2 or 3×3 matrix *(as A above).*

- A matrix may be **transposed** if its columns are swapped over with its rows.

e.g. if $A=\begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 0 & 4 \end{pmatrix}$, the **transpose** of A, A' (or A^T) $=\begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

- In square matrices, if $A' = A$, the matrix, A, is **symmetrical.**

  Also, if $A' = -A$, the matrix, A, is **skew symmetrical.**
  (~~see page 4 of MiA AH3~~).

Simple Matrix Operations

Addition/Subtraction

Only matrices of the same order may be added or subtracted.

This is done by adding subtracting elements in the same position.

Example

$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 6 \\ -1 & 0 \\ -4 & -3 \end{pmatrix}$

$A+B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ -1 & 0 \\ -4 & -3 \end{pmatrix}$

$= \begin{pmatrix} 7 & 9 \\ 0 & 4 \\ -6 & -3 \end{pmatrix}$

Multiplication by a Scalar

Any matrix may be multiplied by a scalar. Simply multiply each element by the scalar.

Example

If $A = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -3 \\ 0 & 1 \end{pmatrix}$, calculate the matrix $2A - 3B$.

$$2A - 3B = 2\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} - 3\begin{pmatrix} 3 & -3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix} - \begin{pmatrix} 9 & -9 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 9 \\ -4 & 1 \end{pmatrix}$$

Exercise **MiA AH p228-230 to read**
MiA AH p231 Ex 13.1 Q 1(a, c, d, j), 3(c), 4(f), 5, 7, 8, 9, 10

"Simple" Matrix Algebra

The following results can be easily shown for matrices :-

- **the commutative law:** $A+B=B+A$, but $A-B \neq B-A$ as we would expect.
- **the associative law:** $(A+B)+C = A+(B+C) = A+B+C$
- **the distributive law for multiplication by a scalar :**

$$k(A+B) = kA + kB$$

- **note also:** $(A')' = A$

  $(A+B)' = A' + B'$

  $(kA)' = kA'$

Exercise: **MiA AH p232 Ex13.2** (Proofs of above !)

Multiplication of Matrices

Remember that matrices can be thought of as tables containing information - much like a spreadsheet in fact.

Imagine that a cinema has three different screens: A, B and C. On a particular day a group of 18 people plan to visit the cinema. The make-up of the group and the screens they plan to visit are shown in the table :

Screen	Screen A	Screen B	Screen C
Adults	3	4	5
Children	2	1	3

This information may be put in a 2×3 matrix as follows : $\begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 3 \end{pmatrix}$

At this cinema the price depends on what time of day you visit, summarised in the table :-

Costs (£)	Adult Ticket	Child Ticket
Morning	4	2
Afternoon	6	3
Evening	8	4

This information can be put in a 3×2 matrix as follows :- $\begin{pmatrix} 4 & 2 \\ 6 & 3 \\ 8 & 4 \end{pmatrix}$

The costs for each screen, depending on what time of day that the group visits, could be calculated from this table :-

Costs (£)	Screen A	Screen B	Screen C
Morning	$4\times 3 + 2\times 2$	$4\times 4 + 2\times 1$	$4\times 5 + 2\times 3$
Afternoon	$6\times 3 + 3\times 2$	$6\times 4 + 3\times 1$	$6\times 5 + 3\times 3$
Evening	$8\times 3 + 4\times 2$	$8\times 4 + 4\times 1$	$8\times 5 + 4\times 3$

We can see this is the combination (multiplication) of the 2 matrices in particular order, prices first:-

row 1 x column 1 row 1 x column 2.

$$\begin{pmatrix} 4 & 2 \\ 6 & 3 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4\times 3+2\times 2 & 4\times 4+2\times 1 & 4\times 5+2\times 3 \\ 6\times 3+3\times 2 & 6\times 4+3\times 1 & 6\times 5+3\times 3 \\ 8\times 3+4\times 2 & 8\times 4+4\times 1 & 8\times 5+4\times 3 \end{pmatrix} = \begin{pmatrix} 16 & 18 & 26 \\ 24 & 27 & 33 \\ 30 & 36 & 52 \end{pmatrix}$$

row 3 x column 1

From the result we can see that multiplying these matrices like this allows us to see the cost for each screen, depending on the time of day, for this group, at a glance.

Care must be taken when multiplying matrices :-

- **order *is* important**
- **"multiply rows into columns"**
- **matrices can only be multiplied together if the first matrix has the same number of columns as the second matrix has rows.**
  Such matrices are said to be **conformable**.

The diagram on below will help you grasp how to go about matrix multiplication.

For multiplying two 3×3 matrices together start as follows :-

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} a\times j+b\times m+c\times p & a\times k+b\times n..... & a\times l+... \\ d\times j+...... & ... & ... \\ g\times & ... & ... \end{pmatrix}$$

In general, if we multiply a $m\times n$ matrix by a $n\times p$ matrix we end up with a $m\times p$ matrix. must match
(See MiA AH page 235 for a nice explanation involving dominoes)

Examples

Calculate the products of the following pairs of matrices :-

1. $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 2+3 & 4+4 \\ 1+9 & 2+12 \end{pmatrix}$

$= \begin{pmatrix} 5 & 8 \\ 10 & 14 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 6+0 & -4+0 \\ 4+0 & -2-1 \\ 12+0 & -6+0 \end{pmatrix}$

$= \begin{pmatrix} 6 & -4 \\ 4 & -3 \\ 12 & -6 \end{pmatrix}$

(3×2) and (2×2) → get 3×2. matrix.

Exercise :- **MiA AH p235 Ex 13.3 Q1(a)(d), 2(a,d,f,g,k,m,p), 3, 4**

Properties of Matrix Multiplication

The following rules hold for matrix multiplication :-

- *In general:* $AB \neq BA$ the commutative law does **not**, therefore, hold.
- $(AB)' = B'A'$ the transpose reversal rule holds.
- $A(BC) = (AB)C$ the associative law holds.
- $A(B+C) = AB + BC$ the distributive law holds.

* order is important.

Identity Matrices, *(I)*

When we multiply a number by 1 (unity) it is unchanged.

For a matrix to remain unchanged on multiplying, we must multiply it by an **identity matrix.**

An identity matrix is one in which all the elements of the leading diagonals are 1 and all other elements are 0.

For a 2×2 matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For a 3×3 matrix, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Orthogonality

A matrix, A, is said to be orthogonal if :- $A'A = I$

Powers of a matrix can often be found as shown in the example below rather than by repeated multiplication

Example

(a) Given that $A = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}$, write A^2 in the form $cA + dI$, stating the values of c and d.

(b) Hence determine A^3 in a similar form and ultimately as a single matrix.

(a)

$$
\begin{aligned}
A^2 &= AA \\
&= \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 2 \\ 1 & 11 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= A + 8I
\end{aligned}
$$

$c = 1 \quad d = 8.$

(b)

$$
\begin{aligned}
A^3 &= AA^2 \\
&= A(A + 8I) \\
&= A^2 + 8AI \\
&= A + 8I + 8A \\
&= 9A + 8I. \\
&= 9\begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \\
&= \begin{pmatrix} -10 & 18 \\ 9 & 35 \end{pmatrix}.
\end{aligned}
$$

Exercise :- MiA AH p236 Ex 13.4 Q 3, 8 + Read the summary

P238 Ex 13.5 Q 1(a), 2(c), 6, 10, 14, 16

Determinants of Square Matrices

By forming an augmented matrix and performing ERO's on the following system of 2 equations in 2 variables:-

$$\begin{aligned} ax+by&=e \\ cx+dy&=f \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$$

we calculate :- $x=\dfrac{de-bf}{ad-bc}$ and $y=\dfrac{af-ce}{ad-bc}$ see MiA p240 (~~see MiA Unit 3, page 15~~)

Obviously solutions for x and y are only possible if $ad-bc\neq 0$

For 2×2 matrix, $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this is known as the **determinant** of matrix A, and is written :-

$$\boxed{\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

✱ ✱ Learn.

Note that :-

- **Only square matrices have determinants**
- **The determinant of a matrix is a *value.***

Exercise:- MiA p240 Ex 13.6 Q 1(a, c, e, h), 2, 3, 4(a, c)

Considering the system of equations :-

$$\begin{aligned} &ax+by+cz=r \\ &dx+ey+fz=s \\ &gx+hy+iz+t \end{aligned}\ , \text{ the corresponding augmented matrix, } A=\left(\begin{array}{ccc:c} a & b & c & r \\ d & e & f & s \\ g & h & i & t \end{array}\right),$$

and using ERO's as before, we can find the determinant of a 3×3 matrix, such that :-

$$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei-fh)-b(di-fg)+c(dh-ge)$$

determinant of matrix left if row and column a is in are deleted

row and column b is in are deleted

signs go + − +

It is probably impractical to remember this, but the following diagram helps calculate the determinant of $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

- **Write out the elements of the matrix twice :-**

$$\begin{matrix} a & b & c & a & b & c \\ d & e & f & d & e & f \\ g & h & i & g & h & i \end{matrix}$$

- **Add each product going down from left to right**
- **Subtract each product going down from right to left**

$$\boxed{\textbf{Thus det } \boldsymbol{A} = aei + bfg + cdh - afh - bdi - ceg}$$

ignore!

Example

Calculate the determinant of the 3×3 matrix, $A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 0 & 4 \\ 2 & 1 & 3 \end{pmatrix}$

$$\det A = 1\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} -3 & 0 \\ 2 & 1 \end{vmatrix}$$

you get a mark for showing this line.

$$= 1(0-4) - 2(-9-8) + 1(-3-0)$$

$$= -4 + 34 - 3$$

$$= 27.$$

Exercise:- **Scholar Unit 3; Section 12.4.1 (or online exercise);**
MiA AH p247 Ex 13.9 Q 4, 6, 7, 8(a)(b)

The Inverse of a 2×2 Square Matrix

Any rational number, $\frac{a}{b}$, has a multiplicative inverse, $\frac{b}{a}$ such that $\frac{a}{b} \times \frac{b}{a} = 1$.

Square matrices have inverses, such that for a matrix, A, and its inverse A^{-1},

$A.\ A^{-1} = I$, the identity matrix.

Remember, for a 2×2 matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

for a 3×3 matrix, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

How do we calculate A^{-1} ?

For a 2×2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

If $\quad A.\ A^{-1} = I$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

hence
$$ap+br = 1 \ldots\ldots\ldots (\underline{1})$$
$$aq+bs = 0 \ldots\ldots\ldots (\underline{2})$$
$$cp+dr = 0 \ldots\ldots\ldots (\underline{3})$$
$$cq+ds = 1 \ldots\ldots\ldots (\underline{4})$$

By re-arranging equation <u>3</u> and substituting into <u>1</u>, and re-arranging equation <u>2</u> and substituting into <u>2</u> **(see AH3, ~~pages 16, 1~~7)** we can work out that :-
p241/242.

$$p = \frac{d}{ad-bc}, \quad q = -\frac{b}{ad-bc}, \quad r = \frac{c}{ad-bc}, \quad p = \frac{a}{ad-bc}$$

thus
$$A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Summary :-

make negative on this diagonal

Learn.

- For a 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

det A. change round numbers on this diagonal.

- $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is called the *adjoint* or the *adjugate* of A and written adj(A).

  Hence $A^{-1} = \dfrac{adj(A)}{\det(A)}$

- note that :- if $ad - bc \neq 0$, thus $\det A \neq 0$ or $|A| \neq 0$, A is said to be *non-singular* and *invertible.*

- A *singular* matrix is one in which $ad\text{-}bc = 0$

Singular matrix $\Rightarrow \det A = 0$.

Example

If $A = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}$, determine the inverse matrix, A^{-1}.

$$\det A = 2 \times 0 - (-3) \times 1$$
$$= 0 + 3$$
$$= 3.$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

Example

For the invertible matrix, B, $B^2 = 3B - I$, prove that :-

(a) $B^3 = 8B - 3I$ and (b) $B^{-1} = 3I - B$.

(a) $B^3 = BB^2$

$= B(3B - I)$

$= 3B^2 - B$

$= 3(3B - I) - B$

$= 9B - 3I - B$

$= 8B - 3I$ as required.

(b) $B^2 = 3B - I$

$B^2 - 3B = -I$

$3B - B^2 = I$

$B(3I - B) = I$

since $BB^{-1} = I$

$B^{-1} = 3I - B$ as required.

Exercise :- MiA AH p243 Ex 13.7 Q 1, 2, 3(a)(g)(k)(q), 5, 6, 8, 10, 11
P244 Ex 13.8 Q 1, 2, 5, 6, 8, 9, 12

The Inverse of a 3×3 Square Matrix

The inverse of a 3×3 matrix can be found as seen previously, $A^{-1}=\dfrac{adj(A)}{\det(A)}$, but in practice this is very difficult to calculate.

It is much easier to use ERO's (as used in Gaussian Elimination) and the identity matrix, I, to convert the augmented matrix $(A|I)$ into the augmented matrix $(I|A^{-1})$.

Use the following procedure :-

- Construct the augmented matrix $\left(A\left|\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right.\right)$
- Use ERO's to reduce matrix A to I.
- Perform the same ERO's on both the left and right parts of the augmented matrix.
- When the left of the augmented matrix is I, the right of the augmented matrix will be A^{-1}.

Example :-

Determine the inverse of the matrix, $A=\begin{pmatrix}4&-1&1\\-1&2&3\\3&-1&1\end{pmatrix}$ and hence solve

the system of equations :-

$$\begin{aligned}4x-y+z&=1\\-x+2y+3z&=0\\3x-y+z&=0\end{aligned}$$

$$\left(\begin{array}{ccc|ccc}4&-1&1&1&0&0\\-1&2&3&0&1&0\\3&-1&1&0&0&1\end{array}\right)$$

get this in form $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ by row operations.

$$\begin{array}{l}\\4R_2+R_1\\4R_3-3R_1\end{array}\left(\begin{array}{ccc|ccc}4&-1&1&1&0&0\\0&7&13&1&4&0\\0&-1&1&-3&0&4\end{array}\right)$$

$$\begin{array}{r} \\ \\ 7R_3 + R_2 \end{array} \left(\begin{array}{ccc|ccc} 4 & -1 & 1 & 1 & 0 & 0 \\ 0 & 7 & 13 & 1 & 4 & 0 \\ 0 & 0 & 20 & -20 & 4 & 28 \end{array}\right)$$

$$\begin{array}{r} 20R_1 - R_3 \\ 20R_2 - 13R_3 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 80 & -20 & 0 & 40 & -4 & -28 \\ 0 & 140 & 0 & 280 & 28 & -364 \\ 0 & 0 & 20 & -20 & 4 & 28 \end{array}\right)$$

$$\begin{array}{r} 7R_1 + R_2 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 560 & 0 & 0 & 560 & 0 & -560 \\ 0 & 140 & 0 & 280 & 28 & -364 \\ 0 & 0 & 20 & -20 & 4 & 28 \end{array}\right)$$

$$\begin{array}{r} \frac{1}{560}R_1 \\ \frac{1}{140}R_2 \\ \frac{1}{20}R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & \frac{1}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & -1 & \frac{1}{5} & \frac{7}{5} \end{array}\right)$$

$$\text{so } A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & \frac{1}{5} & -\frac{13}{5} \\ -1 & \frac{1}{5} & \frac{7}{5} \end{pmatrix}$$

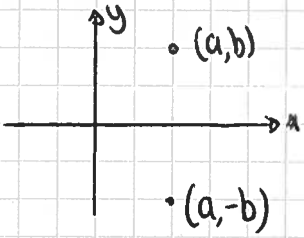
<u>Check</u>

$$AA^{-1} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 2 & 3 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & \frac{1}{5} & -\frac{13}{5} \\ -1 & \frac{1}{5} & \frac{7}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Transformations Using 2x2 Matrices.

① Reflection in x-axis



point $(a, b) \rightarrow (a, -b)$

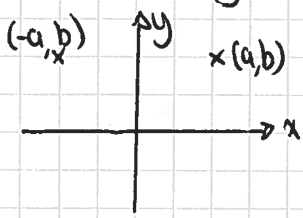
Using matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

Matrix for reflection in x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Points on x-axis are invariant (stay the same)

② Reflection in y-axis



$(a, b) \rightarrow (-a, b)$

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{matrix}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$$

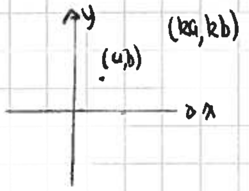
③ ~~Reflection~~ In a similar way we can find the matrices for the following transformations

* You need to learn these or be able to deduce them *

Transformation	Mapping	Matrix
reflection in x-axis	$(a, b) \rightarrow (a, -b)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in y-axis	$(a, b) \rightarrow (-a, b)$	$\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$
reflection in $y = x$	$(a, b) \rightarrow (b, a)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection in $y = -x$	$(a, b) \rightarrow (-b, -a)$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
half turn about O	$(a, b) \rightarrow (-a, -b)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
rotation of 90° about O	$(a, b) \rightarrow (-b, a)$	$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$

Rotation $-90°$ about O $\quad (a,b) \to (b,-a) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Dilatation $[0, k]$ $\quad (a,b) \to (ka, kb) \quad \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.



* Rotation of θ radians anticlockwise about O *

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

* given on formula sheet. *

Examples

① A, B and C are the points $(1,0)$ $(1,1)$ and $(0,1)$. Calculate the co-ordinates of their images under transformation T with matrix $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

* Just pre-multiply each point by the matrix to get its image

$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad$ A' is $(2,-1)$

$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\quad$ B' is $(3,1)$

$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad$ C' is $(1,2)$

② Find the 2×2 matrix which will transform the point $(-3,2)$ to $(-8,-13)$ and the point $(5,4)$ to $(6,7)$.

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$(-3, 2) \rightarrow (-8, -13)$ $\Rightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -13 \end{pmatrix}$

$-3a + 2b = -8$ ①

$-3c + 2d = -13$ ②

Also $(5, 4) \rightarrow (6, 7)$ $\Rightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

~~5a~~ $5a + 4b = 6$ ③

$5c + 4d = 7$ ④

Solving ① and ③

$-6a + 4b = -16$

$5a + 4b = 6$

$11a = 22$

$a = 2$

$\Rightarrow b = -1$

Solving ② and ④

$-3c + 2d = -13$

$5c + 4d = 7$

$-6c + 4d = -26$

$5c + 4d = 7$

$11c = 33$

$c = 3 \Rightarrow d = -2.$

Matrix $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$

Maths in Action

p 257 Exercise 13.10

Compos

③ Find the invariant points of the transformation $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$

points which stay the same

so $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$2x + y = x \Rightarrow y = -x$

$x + 2y = y \quad y = -x$

subtract ~~x - y = x - y~~

Points on line $y = -x.$

p253 Exercise 13.11

Q.①

Composition of Two Transformations.

Here we have 2 transformations T_1 and T_2 one after the other. To get the matrix for the combined transformation we just multiply the two matrices in the correct order.

T_1 followed by T_2 is given by $\begin{pmatrix} \text{matrix for} \\ T_2 \end{pmatrix} \begin{pmatrix} \text{matrix for} \\ T_1 \end{pmatrix}$

Example.

Find the matrix associated with a reflection in $y=x$ followed by a reflection in the x-axis

matrix $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{reflection in } x\text{-axis}} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{reflection in } y=x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

p255 Ex 13.12

combined we have a rotation of $-90°$ about O.

Inverse Transformations.

If we want to find a matrix which 'undoes' a certain transformation we just find the inverse of the matrix.

(since $A^{-1}A = I$)

Note For orthogonal matrices $A^T = A^{-1}$ so A^T will map the image under A back to the original point.

(The matrices for reflection and rotation are orthogonal).

~~Images of Lines and Circles~~.

Image of a Curve.

Find the equation of the image of the curve with equation $y=3x^2$ under the transformation with associated matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

Let (a, b) on curve $\rightarrow (x, y)$

We know $b = 3a^2$

Also $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{-1}{1}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

inverse.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3x + 2y \\ 2x - y \end{pmatrix}$$

$a = 2y - 3x$

$b = 2x - y$

Substitute into $b = 3a^2$

$$2x - y = 3(2y - 3x)^2.$$

p253 Exercise 13.11

Q 2

~~**Exercise :- MiA AH3; page 28; Exercise 8**~~

Transformation Matrices

A point, P (x, y) on the coordinate plane may be *transformed* in several different ways:-

- **Translation :-** a slide across the plane
- **Rotation**
- **Reflection**
- **Dilatation :-** a scaling

Given that the point P (x, y) can be represented by the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, and that under a linear transformation $P(x, y) \rightarrow P'(ax+by, cx+dy) = P'(x', y')$, we can see that transformations may be represented by matrices :-

$$\begin{aligned} x' &= ax+by \\ y' &= cx+dy \end{aligned} \quad \text{becomes} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$ which shows the position vector of P' is obtained by pre-multiplying the position vector of P by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

In General

Example :-

Determine the image points of the triangle with vertices A(1, 1); B(2, 6) and C(-3, 0) under the transformation associated with the matrix $\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$.

$A(1,1)$ $\quad \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad A'(2,1)$

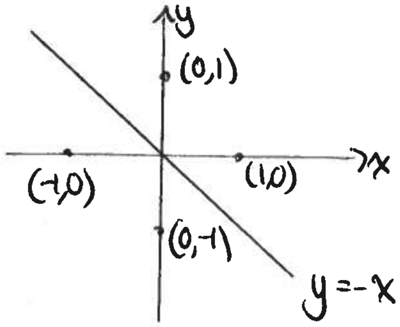
$B(2,6)$ $\quad \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\quad B'(0,2)$

$C(-3,0)$ $\quad \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \end{pmatrix}$ $\quad C'(-9,-3)$

A transformation matrix can be constructed by finding the images of the 2 points with co-ordinates (1,0) and (0, 1) – see **MiA AH3, ~~page 30~~** 250

Example :- Construct a transformation matrix for the reflection in the line $y = -x$ and hence find the image of the point P(-2, -3).

$(1,0) \to (0,-1)$
$(0,1) \to (-1,0)$



matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$a = 0 \quad c = -1$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$b = -1 \quad d = 0$

matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$P(-2,-3) \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\underline{P'(3,2)}$

Exercise :- MiA AH3; page 32, Exercise 9A, Q1-6.

<u>Rotation (about the Origin) Matrix</u>

For any rotation of a point P(*x*, *y*) of angle θ about O, the origin, the matrix :-
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is used as a pre-multiplier.

<u>Reflection Matrix</u>

For reflecting a point P(*x*, *y*) in a line through the origin, which makes an angle of θ, $-90° \le \theta \le 90°$, the matrix :- $\begin{pmatrix} \cos 2\theta & \sin 2\vartheta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ is used as a pre-multiplier.

<u>Dilatation Matrix</u>

The point P(*x*, *y*) may be "scaled" by pre-multiplying by the matrix :- $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

Exercise :- Scholar, Unit 3, Section 12.6 – do all examples including the rotation, reflection, dilatation and general exercise

Composition of Transformations

Transformations can, of course, be combined and consequently a matrix for the resultant composite transformation can be constructed.

In general, order is important.

Example

1. Construct a single matrix AB for the transformation associated with reflection in the line $y = -x$, followed by an anti-clockwise rotation of $\frac{\pi}{4}$ about the origin.

2. Repeat 1. for a matrix BA associated with the same transformations in the reverse order.

3. Find the co-ordinates of the image of the point (1, 3) under each combined transformation.

① From previous example reflection in line $y = -x$ has matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

anticlockwise rotation of $\frac{\pi}{4}$ $\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

② $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

~~Exercise :- MIA AH3; page 39; Exercise 9B; Q 1 to 4~~.

In ① $P(1,3)$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}, & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2\sqrt{2}}{2} \\ -2\sqrt{2} \end{pmatrix} \qquad P'\left(-\frac{2\sqrt{2}}{2}, -2\sqrt{2}\right)$$

In ② $P(1,3)$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -2\sqrt{2} \\ \frac{2\sqrt{2}}{2} \end{pmatrix} \qquad P'\left(-2\sqrt{2}, \frac{2\sqrt{2}}{2}\right)$$