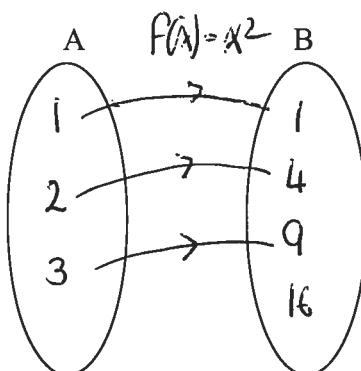


FUNCTIONS

Revision

A function from set A to set B maps each member of A with exactly one member of B.

Eg

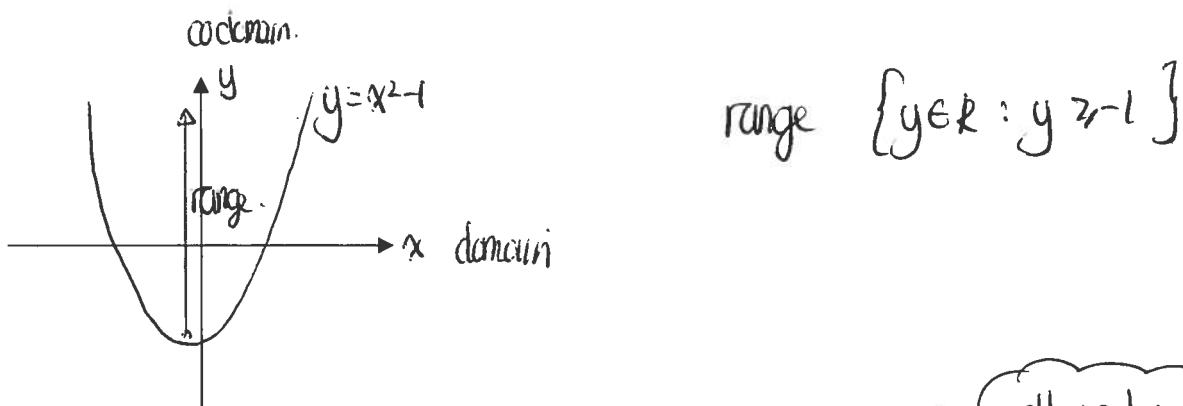


Set A is called the ...domain...

Set B is called the....codomain

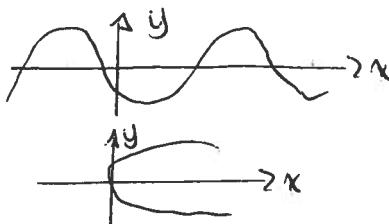
The range is $\{1, 4, 9\}$ (ie the values of $f(x)$)

Most functions have the set of real numbers R for their domain and codomain. They can be represented graphically.



Note

For a graph of a function each x co-ordinate is related to one and only one y co-ordinate.



is the graph of a function

is not the graph of a function

Example

What is the largest possible domain for the following functions ?

$$(a) f(x) = \sqrt{(x+2)}$$

$$(b) f(x) = \frac{1}{x^2 - 9}$$

(a) We can't find the square root of a negative number

$$\text{so } x+2 \geq 0$$

$$x \geq -2$$

$$\text{domain } \{ x \in \mathbb{R} : x \geq -2 \}$$

(b) We can't have zero or the bottom of a fraction

$$\text{so } x^2 - 9 \neq 0$$

$$x^2 \neq 9$$

$$x \neq \pm 3$$

$$\text{domain } \{ x \in \mathbb{R} : x \neq \pm 3 \}$$

Tips for success

Remember

- domain refers to x numbers
- codomain / range refer to y numbers
- to find range - draw graph and read off y values.

Learning about

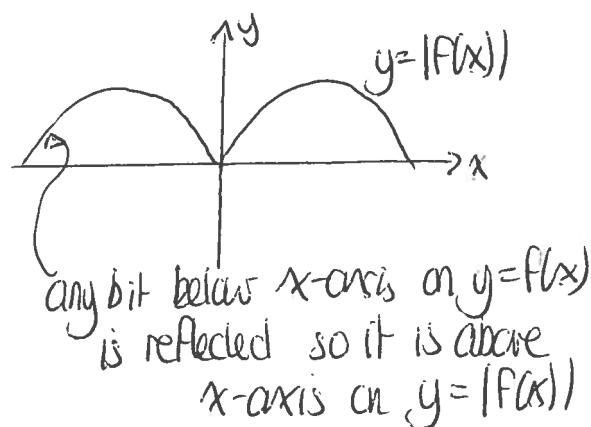
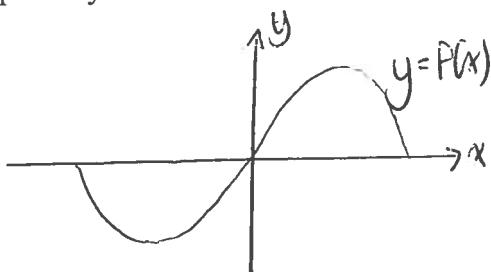
The Modulus Function

$|x|$ means the magnitude of x or modulus of x .

We ignore the sign of x .

$$|7| = 7 \quad |-7| = 7$$

Graphically -



Tips for success

- taking modulus means anything positive stays the same and anything negative changes sign to become positive
- to draw graph $y = |f(x)|$ leave all bits above x-axis same and reflect all bits below x-axis so they are above x-axis.

Learning to make simple

Simple Transformations

1. $y = f(x) + c$ is $y = f(x)$ translated (moved) a distance c parallel to the y -axis.
2. $y = f(x - c)$ is $y = f(x)$ translated a distance c parallel to the x – axis
eg $y = f(x - 2)$ moves $f(x)$ 2 units to the right
 $y = f(x + 2)$ moves $f(x)$ 2 units to the left.
3. $y = -f(x)$ is $y = f(x)$ reflected in the x – axis.
4. $y = f(-x)$ is $y = f(x)$ reflected in the y – axis.
5. $y = f(ax)$ is $y = f(x)$ squashed up a times. in x -direction ie $(p, q) \rightarrow (\frac{p}{a}, q)$
6. $y = af(x)$ is $y = f(x)$ stretched a times in y direction
ie $(p, q) \rightarrow (p, aq)$

Note Numbers on 'outside' of $f(x)$ change y -co-ords
(eg $kf(x)$, $f(x)+k$)

Numbers inside bracket with x ie $f(kx)$, $f(x+k)$
change x co-ords (opposite of what you
might think)

Tips for success

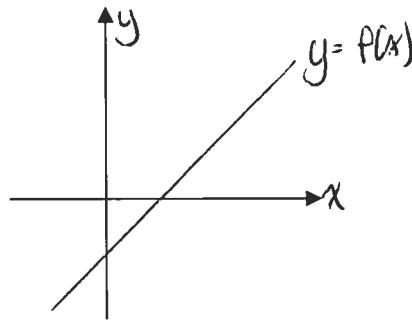
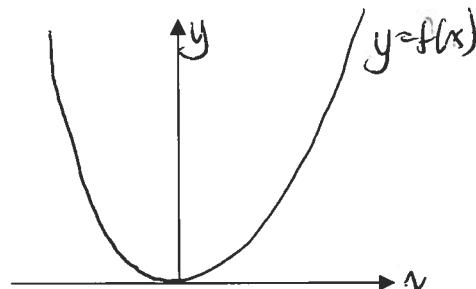
- to draw new graph think about what transformations are being performed then work out new co-ords of any given points.
- think about new position of points where graph cuts axes.
- make sure any co-ords on new graph are marked on .

Learning about

Inverse Functions $y = f^{-1}(x)$

For a function from set A to set B to have an inverse the sets A and B must be in one to one correspondence ie. each member of A corresponds to one member of B and each member of B corresponds to one member of A.

eg

 $f^{-1}(x)$ exists $f^{-1}(x)$ does not exist
for domain \mathbb{R} .

$f^{-1}(x)$ exists if domain and co-domain are restricted to
 $\{x, y \in \mathbb{R} : x, y > 0\}$

Finding a Formula for an Inverse Function

If $y = f(x)$ then $f^{-1}(y) = x$

To find an inverse function let $y = f(x)$ and rearrange to make x the subject.

Example

$$f(x) = \frac{x}{x+3}$$

Find the inverse function.

If $y = f(x)$ then $f^{-1}(y) = x$

Here $y = \frac{x}{x+3}$

* change subject to x

$$(x+3)y = x$$

$$xy + 3y = x$$

$$xy - x = -3y$$

multiply out

get all x 's together

$$\begin{aligned} x(y-1) &= -3y \\ x &= \frac{-3y}{y-1} \\ y-1 & \end{aligned}$$

so $f^{-1}(y) = \frac{-3y}{y-1} = \frac{3y}{1-y}$ (since $f^{-1}(y) = x$)

$$f^{-1}(x) = \frac{3x}{1-x}$$

change all letters to x

Graphs of Inverse Functions

The graph of $y = f^{-1}(x)$ is a reflection of $y = f(x)$ in the line $y = x$

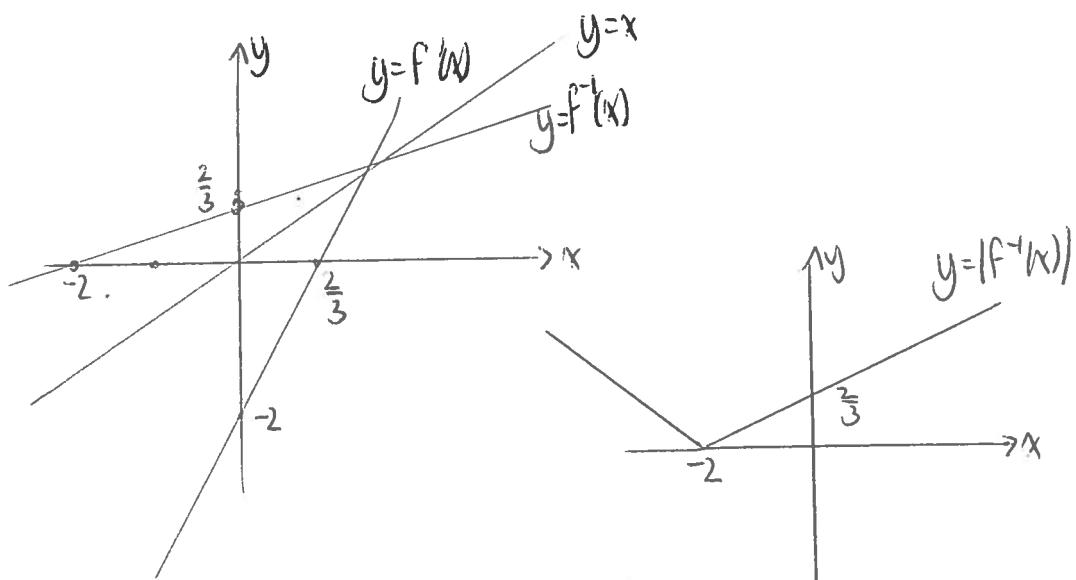
Note domain of $f = \text{range of } f^{-1}$

Range of $f = \text{domain of } f^{-1}$

Example

Consider $f(x) = 3x - 2$. Sketch the graph of $y = |f^{-1}(x)|$

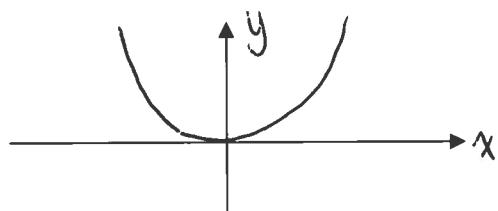
$$\begin{array}{lll} y = 3x - 2 & \text{Straight line} & \\ \text{cuts } x\text{-axis} & y=0 & x = \frac{2}{3} \quad (\frac{2}{3}, 0) \\ \text{cuts } y\text{-axis} & x=0 & y=-2 \quad (0, -2) \end{array}$$



Sometimes functions can only have an inverse if we restrict the domain.

Example

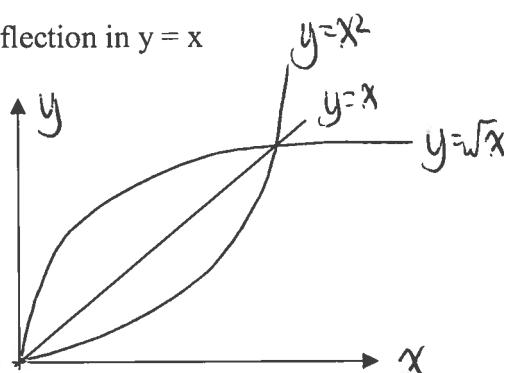
$y = x^2$
The function



$y = x^2$ only has an inverse function
if either $x \geq 0$
or $x \leq 0$

If We choose domain $x > 0$ function (graph) has an inverse

Graph of inverse function is a reflection in $y = x$



Note

Maths in Action Book 1 Page 100 Exercise 3

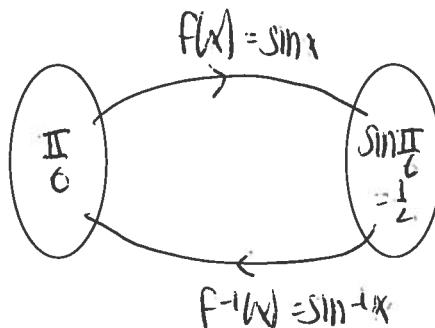
CfE p67 Exercise 5.3

Inverse Trig Functions

$$f : x \rightarrow \sin x$$

$$f^{-1} : \sin x \rightarrow x$$

eg

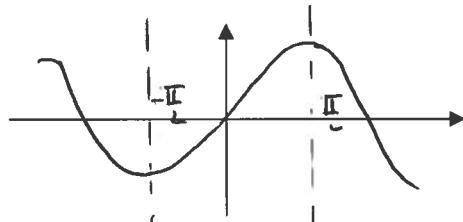


If $f(x) = \sin x$
 then $f^{-1}(x) = \sin^{-1}x$

Graphs

Inverse Sine

Sine function



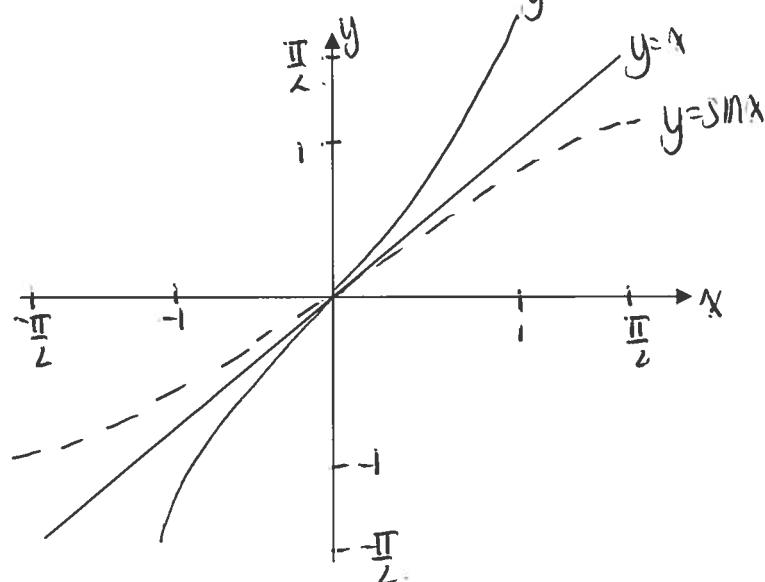
The sine function is
not in one to one
 correspondence.

This function can only have an inverse if we restrict the domain.

We choose

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for the domain

Graph of inverse function is a reflection in $y = x$, $y = \sin^{-1}x$



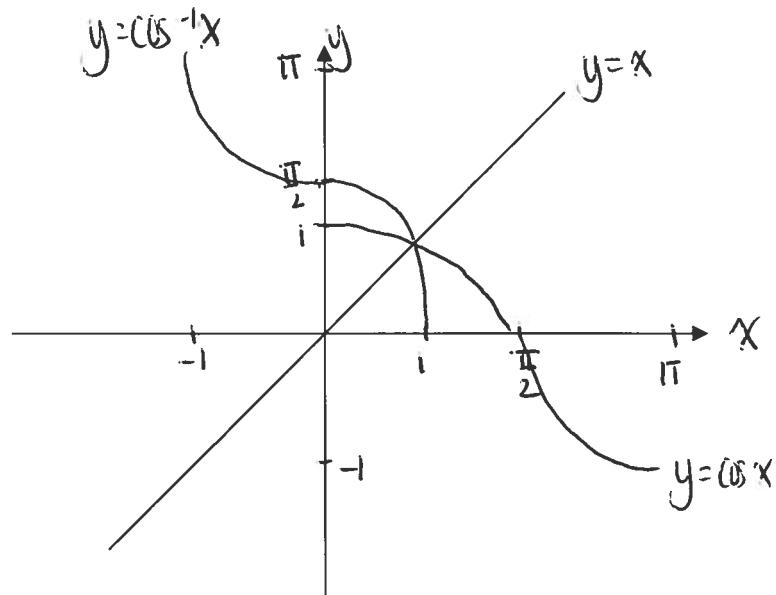
Note

For $f(x) = \sin^{-1}x = \theta \Leftrightarrow$ angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ inclusive
 ie 1st or 4th quadrant

x between
 -1 and 1

Inverse Cosine

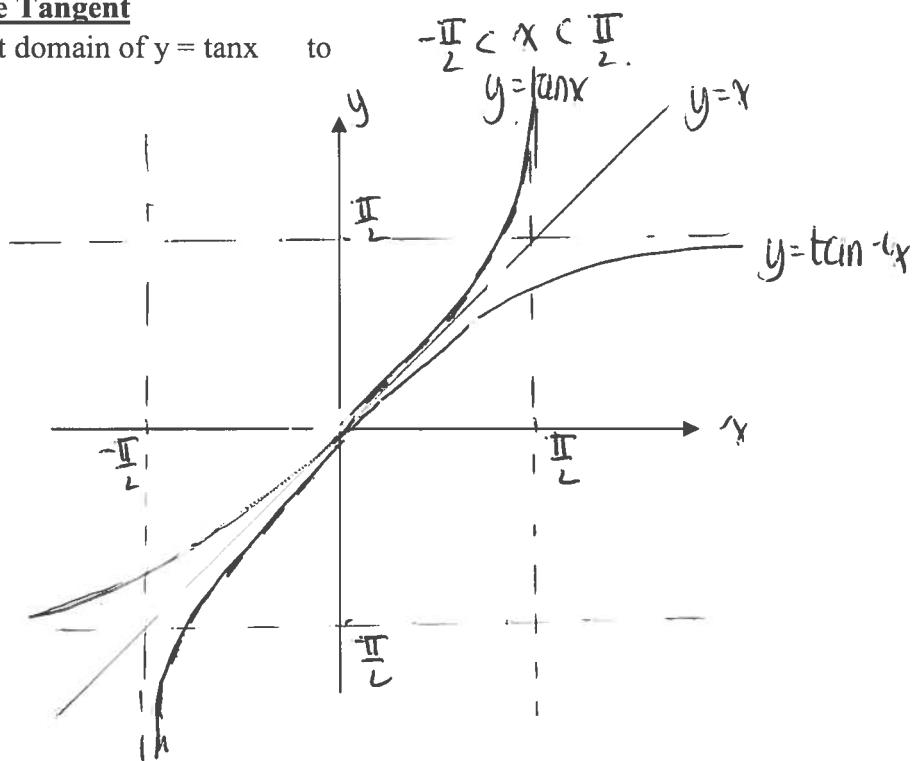
Restrict the domain of $y = \cos x$ to $0 \leq x \leq \pi$

**Note**

$f(x) = \cos^{-1} x = \theta$ \rightarrow answer between 0 and π inclusive
 x can be between -1 and 1 inclusive
 1st or 2nd quadrant

Inverse Tangent

Restrict domain of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



Note

$f(x) = \tan^{-1} x = \theta$

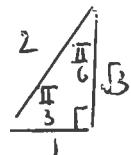
x can be any real number

angle must be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ not inclusive

* * learn

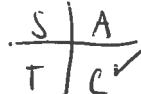
Examples

$$\begin{aligned} \textcircled{1} \quad \sin^{-1} \frac{1}{2} \\ &= \frac{\pi}{6} \end{aligned}$$



$$\begin{aligned} \textcircled{2} \quad \sin^{-1} \left(-\frac{1}{2}\right) \\ &= -\frac{\pi}{6} \end{aligned}$$

in $\frac{\pi}{6}$ angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$
(1st or 4th)



$$\textcircled{3} \quad \cos^{-1}(4) \text{ not possible.}$$

$$\begin{aligned} \textcircled{4} \quad \cos^{-1} \left(-\frac{1}{2}\right) \\ &= \frac{2\pi}{3} \end{aligned}$$

in $\frac{\pi}{3}$ angle between 0 and π
1st or 2nd



$$\begin{aligned} \textcircled{5} \quad \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right) \\ &= -\frac{\pi}{6} \end{aligned}$$

in $\frac{\pi}{6}$
 \tan^{-1} 1st or 4th.

$$\begin{aligned} \textcircled{6} \quad \sin^{-1} \left(\sin \frac{\pi}{6}\right) \\ &= \sin^{-1} \frac{1}{2} \\ &= \frac{\pi}{6} \end{aligned}$$

Tips

• learn ranges:

$\sin^{-1} x$ 1st or 4th $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\cos^{-1} x$ 1st or 2nd $0 \leq \theta \leq \pi$

$\tan^{-1} x$ 1st or 4th $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned} \textcircled{7} \quad \sin^{-1} \left(\sin \frac{3\pi}{2}\right) \\ &= \sin^{-1} 1 \\ &= \frac{\pi}{2} \end{aligned}$$

• remember $\sin^{-1} (\sin x) \neq x$ for all values etc.

Even and Odd Functions

Even Functions

$f(x)$ is an even function if

$$f(-x) = f(x)$$

Example

$$g(x) = 4x^2 - 3$$

Replace

$$\begin{aligned} &x \text{ by } (-x) \\ g(-x) &= 4(-x)^2 - 3 \\ &= 4x^2 - 3 \\ &= g(x) \text{ so } g(x) \text{ is even.} \end{aligned}$$

Graphs of even functions are

symmetrical about the y-axis



Odd Functions

$f(x)$ is an odd function if

$$f(-x) = -f(x)$$

Example

Show that $h(x) = \frac{x^3 - x}{4}$ is an odd function.

$$\begin{aligned} h(-x) &= \frac{(-x)^3 - (-x)}{4} \\ &= \frac{-x^3 + x}{4} \\ &= -\frac{(x^3 - x)}{4} = -h(x) \end{aligned}$$

so $h(x)$ is an odd function

Graphs of odd functions have

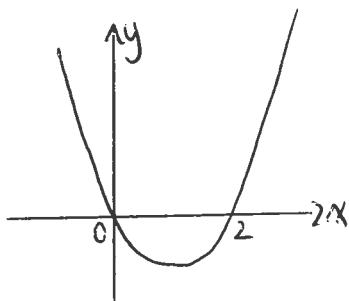
half turn symmetry about 0

Tips for success

- To test if a function is even or odd put $-x$ instead of x .
 - EVEN \rightarrow get same function
 - ODD \rightarrow get negative of function.
- Only some functions are even or odd.
- $\sin(-x) = -\sin x$ odd $\cos(-x) = \cos x$ even.
- Given part of a graph if it is known it is ~~even~~ Even - complete by reflecting in, y -axis
ODD - complete by rotating 180° about 0.

Example

The diagram below shows part of a graph of $f(x)$.
 State whether $f(x)$ is odd, even or neither. [2010]



Not symmetrical about y -axis
 No half turn symmetry about 0.
 So neither odd or even.

Stationary Points and Their Nature

For Stationary Points $\frac{dy}{dx} = 0$

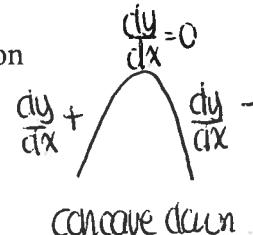
Nature

- Use nature tables (as in Higher)
- Or use second derivative (usually quicker)

Maximum Turning point

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0$$

Explanation

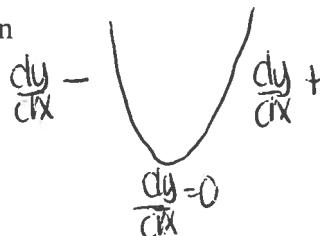


$\frac{dy}{dx}$ is decreasing
 $\Rightarrow \frac{d}{dx}(\frac{dy}{dx}) < 0$
 i.e. $\frac{d^2y}{dx^2} < 0$

Minimum Turning point

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0$$

Explanation

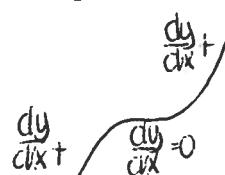


$\frac{dy}{dx}$ is increasing
 i.e. $\frac{d^2y}{dx^2} > 0$

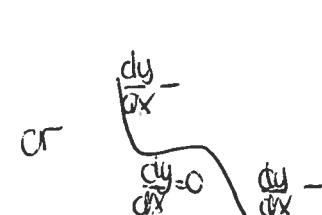
If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ then the stationary point could be a maximum, minimum or a stationary point of inflection.

For point of inflection $\frac{d^2y}{dx^2}$ must change sign too.

Explanation



$\frac{dy}{dx}$ decreases to 0 then increases
 $\Rightarrow \frac{dy}{dx^2}$ changes sign.



$\frac{dy}{dx}$ increases to 0 then decreases
 $\Rightarrow \frac{dy}{dx^2}$ changes sign.

2. Find the stationary point on $f(x) = x^5$ and its nature.

$$f'(x) = 5x^4$$

For stationary points $f'(x) = 0$

$$\begin{aligned} 5x^4 &= 0 \\ x &= 0 \\ \Rightarrow y &= 0 \end{aligned}$$

Nature $f''(x) = 20x^3$

When $x=0$ $f''(x) = 0$

x	\rightarrow	0	\rightarrow
$f''(x)$		-20	0 20

change of sign so $(0,0)$ is

a point of inflection
inflexion
both spelling used

or use $f'(x)$ to get shape.

x	\rightarrow	0	\rightarrow
$f'(x)$		5	0 5

$(0,0)$ is a rising point
of inflection

Examples

1 Find stationary points and their nature for the curve

$$y = x^3 + 3x^2 - 9x + 6$$

$$\frac{dy}{dx} = 3x^2 + 6x - 9$$

Stationary points when $\frac{dy}{dx} = 0$

$$3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x - 3) = 0$$

$$3(x+3)(x-1) = 0$$

$$x = -3 \text{ or } x = 1$$

$$y = 33 \quad y = 1$$

Nature

$$\frac{d^2y}{dx^2} = 6x + 6$$

When $x = -3$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -18 + 6 \\ &= -12\end{aligned}$$

$$< 0$$

so

$(-3, 33)$ is a max TP

When $x = 1$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 12 \\ &> 0\end{aligned}$$

so

$(1, 1)$ is a min TP

More About Points of Inflection

(or inflexion)

We have already considered stationary points of inflection

- $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$
- Use table for $\frac{dy}{dx}$ to check shape.

or look for change of sign
for $\frac{dy}{dx^2}$.

It is also possible to get oblique points of inflection.

For oblique points of inflection

$$\frac{dy}{dx} \neq 0 \text{ and } \frac{d^2y}{dx^2} = 0$$

or does not exist

not equal

Example

Find the coordinates of the point of inflection on the graph $y = x^3 + x$.

could be stationary or oblique.

 $\frac{dy}{dx} = 0$ for all points of inflection

$$y = x^3 + x$$

$$\frac{dy}{dx} = 3x^2 + 1$$

$$\frac{d^2y}{dx^2} = 6x$$

Point of inflection

$$\frac{d^2y}{dx^2} = 0$$

$$6x = 0$$

$$x = 0$$

$$\Rightarrow y = 0 \quad (0, 0)$$

Note

At $(0, 0)$ $\frac{dy}{dx} = 1$ so this is an oblique point of inflection

Check for change of sign.

x	→	0	↓
$\frac{d^2y}{dx^2}$	-6	0	6

Change of sign ✓ so $(0, 0)$ is point of inflection!

2. Show that there is a point of inflection on the graph $y = 2x^{\frac{1}{3}}$ at $x = 0$.

$$y = 2x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{2}{3}}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{9}x^{-\frac{5}{3}}$$

undefined at $x=0$

undefined at $x=0$

* For point of inflection show that $\frac{d^2y}{dx^2}$ changes sign

$$\frac{d^2y}{dx^2} = -\frac{4}{9\sqrt[3]{x^5}}$$

When $x < 0$

$$\Rightarrow \sqrt[3]{x} < 0 \Rightarrow (\sqrt[3]{x})^5 < 0 \Rightarrow \frac{d^2y}{dx^2} \text{ is positive.}$$

When $x > 0$

$$\sqrt[3]{x} > 0 \Rightarrow (\sqrt[3]{x})^5 > 0 \Rightarrow \frac{d^2y}{dx^2} \text{ is negative.}$$

Change of sign so $x=0$ is a point of inflection (inflection)

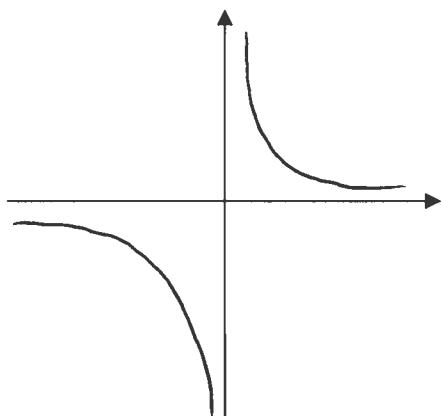
CfE p73 EX 5.7 (5)

Asymptotes

Graphs of rational functions have asymptotes.

Consider the curve $y = \frac{1}{x}$

x	-5	-3	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	3	5
$y = \frac{1}{x}$	$-\frac{1}{5}$	$-\frac{1}{3}$	-1	-2	-4	0	4	2	1	$\frac{1}{3}$	$\frac{1}{5}$



The x and y axes are **asymptotes**.

The curve gets closer and closer to these lines but never touches them.

* Graphs of rational functions

have asymptotes

i.e. with x's in bottom of fractions.

There are three types of asymptotes.

Type 1 Vertical Asymptotes

These occur when the denominator is zero



Example

1. $y = \frac{x^2 - x}{x + 1}$

Vertical asymptote when $x + 1 = 0$

$$x = -1$$

helps for graph sketching later

approach

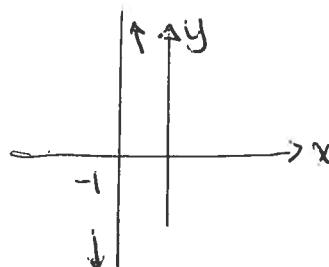
$$\begin{aligned} y &= \frac{x^2 - x}{x + 1} \\ &= \frac{x(x-1)}{(x+1)} \end{aligned}$$

factors

$y = \frac{x}{x(x-1)}$	\rightarrow	-1	\rightarrow
	(-)(-)	∞	(-)(+)
overall	-	∞	+

graph goes down

graph goes up.



2. Find the vertical asymptotes and approach for $y = \frac{3}{x^2 - 3x + 2}$

$$y = \frac{3}{(x-2)(x-1)}$$

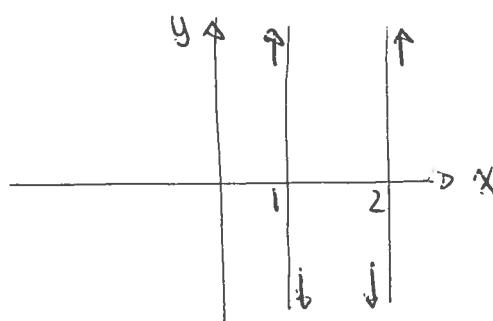
vertical asymptotes $(x-1)(x-2)=0$
 $x=1$ and $x=2$

approach

* Must use separate tables for each asymptote

x	\rightarrow	1	\rightarrow
y	$\begin{pmatrix} + \\ (-) \end{pmatrix}$	∞	$\begin{pmatrix} + \\ (-) \end{pmatrix}$
	+	∞	-
	\uparrow	\downarrow	

x	\rightarrow	2	\rightarrow
y	$\begin{pmatrix} + \\ (-) \end{pmatrix}$	∞	$\begin{pmatrix} + \\ (-) \end{pmatrix}$
	-	∞	+
	\downarrow	\uparrow	\downarrow



Type 2 Horizontal Asymptotes**Type 3 Oblique Asymptotes**

To find either horizontal or oblique asymptotes divide out the function first if the numerator is the same or higher degree than the denominator, and then consider what happens as $x \rightarrow \infty$

Examples

Find horizontal or oblique asymptotes and their approach for the following functions

1. $y = \frac{3x^2}{x^2 + 1}$ * Power on top same or higher degree as bottom
 \rightarrow DIVIDE OUT FIRST

$$\begin{array}{r} x^2+1 | \overline{3x^2} \\ \underline{3x^2+3} \\ -3 \end{array}$$

So $y = 3 - \frac{3}{x^2+1}$

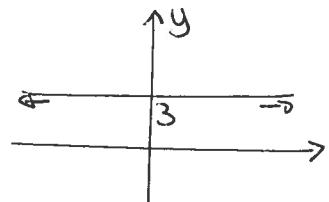
As $x \rightarrow \pm\infty$ $y \rightarrow 3$

so $y=3$ is a horizontal asymptote

Approach $y = 3 - \frac{3}{x^2+1}$

When $x \rightarrow \infty$ $y = 3 - \text{a little bit}$
 (curve below)

$x \rightarrow -\infty$ $y = 3 - \text{a little bit}$
 (curve below)



2. $y = \frac{x^2 + x - 2}{x}$

$$\begin{array}{r} x | \overline{x^2+x-2} \\ \underline{x^2} \\ \underline{x-2} \\ x \end{array}$$

$y = x + 1 - \frac{2}{x}$

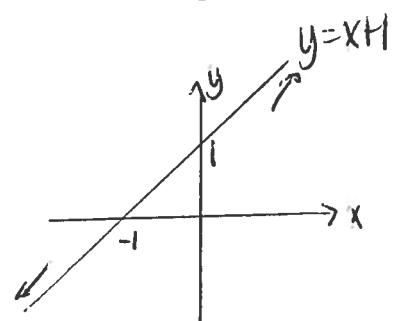
As $x \rightarrow \pm\infty$ $y \rightarrow x + 1$

so $y = x + 1$ is an oblique asymptote

Approach $y = x + 1 - \frac{2}{x}$

As $x \rightarrow \infty$ $y = x + 1 - \text{a little bit}$
 (below)

$x \rightarrow -\infty$ $y = x + 1 + \text{a little bit}$
 (above)



Note

A curve can cut a horizontal or oblique asymptote but not a vertical asymptote

\uparrow
only gives behaviour
as $x \rightarrow \pm\infty$

\nwarrow value of x when $f(x)$
is not defined.

A curve only has one horizontal or oblique asymptote.

Tips for success

- vertical asymptotes
 - make denominator = zero
 - find approach (only if asked to sketch graph)
- horizontal asymptote - divide out if top same or higher degree than bottom
- consider $x \rightarrow \infty$, give equation as $y =$
- find approach to help if sketching graph.

$$\text{eg } y = \frac{3}{x^2+5} \quad \text{DON'T divide out.}$$

Here as $x \rightarrow \infty$, $y \rightarrow 0$
 $y=0$ is horizontal asymptote

Maths in Action Unit 1 p110 Exercise 10

CfE p76 Ex 5.10 (1).

Learning about

Sketching Rational Functions

- Method
1. Find where the curve cuts the x-axis and y-axis
 2. Find any vertical asymptotes and approach.
 3. Find any horizontal or oblique asymptote and approach.
 4. Find stationary points and nature.
 5. Sketch

Example

Sketch $y = \frac{x^2 + 4x + 7}{x+1}$ showing asymptotes and stationary points.

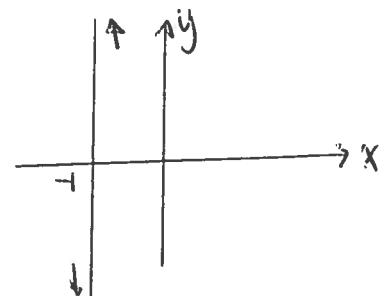
Cuts x-axis $y=0$ $x^2 + 4x + 7 = 0$
 $b^2 - 4ac = 16 - 28$
= negative so no real roots
→ doesn't cut x-axis.

Cuts y-axis $x=0, y=7$ $(0, 7)$

Vertical Asymptotes $x+1=0$
 $\underline{x=-1}$

Approach

x	\rightarrow	-1	\rightarrow
$y = \frac{x^2 + 4x + 7}{x+1}$		$(+)\over(-)$	$(+)\over(+)$
		$-\infty$	$+\infty$



Horizontal / Oblique asymptotes.

$$\begin{array}{r} x+1 \quad | \quad x^2 + 4x + 7 \\ \quad \quad \quad | \quad x^2 + x \\ \quad \quad \quad | \quad 3x + 7 \\ \quad \quad \quad | \quad 3x + 3 \\ \quad \quad \quad | \quad 4 \end{array}$$

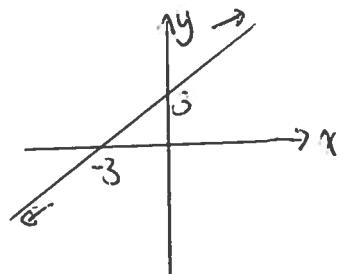
so $y = x+3 + \frac{4}{x+1}$

As $x \rightarrow \pm\infty$ $y \rightarrow x+3$.

so $y = x+3$ is an oblique asymptote

Approach

as $x \rightarrow \infty$ $y \rightarrow x+3 + \text{a little bit}$
 $x \rightarrow -\infty$ $y \rightarrow x+3 - \text{a little bit}$



Stationary Points

$$y = x+3 + \frac{4}{x+1}$$

$$= x+3 + 4(x+1)^{-1}$$

$$\frac{dy}{dx} = 1 - 4(x+1)^{-2}$$

$$= 1 - \frac{4}{(x+1)^2}$$

For stationary points

* Hint → use divided diff form
to find $\frac{dy}{dx}$ → it is much easier.

$$\frac{dy}{dx} = 0$$

$$1 - \frac{4}{(x+1)^2} = 0$$

$$\frac{4}{(x+1)^2} = 1$$

$$4 = (x+1)^2$$

$$x+1 = \pm 2$$

$$x = -3 \quad \text{or} \quad x = 1$$

$$y = -2 \quad \text{or} \quad y = 6$$

points $(-3, -2)$ $(1, 6)$

Nature

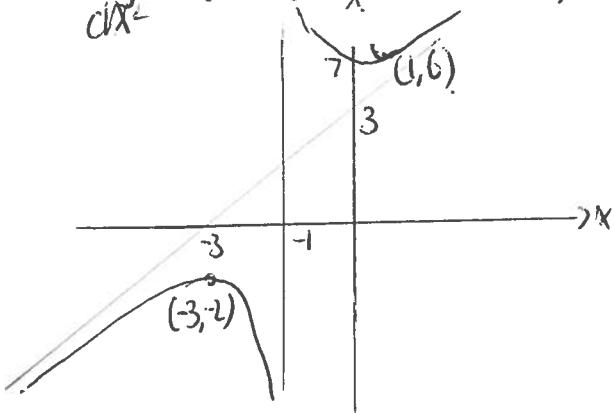
$$\frac{dy}{dx} = 1 - 4(x+1)^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 8(x+1)^{-3}$$

$$= \frac{8}{(x+1)^3}$$

When $x = -3$ $\frac{d^2y}{dx^2} = -1 < 0$ so $(-3, -2)$ is a maximum TP.

When $x = 1$ $\frac{d^2y}{dx^2} = 1 > 0$ so $(1, 6)$ is a minimum TP.



Example

Sketch the function $y = \frac{x^2}{(x+2)(x-1)}$ $x \neq -2, 1$

Cuts x -axis $y=0 \Rightarrow x^2=0$ $x=0$ $(0,0)$

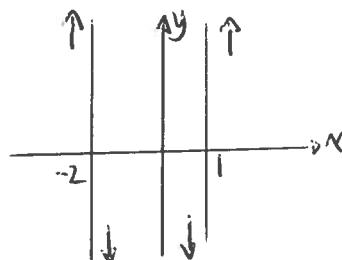
Cuts y -axis $x=0 \Rightarrow y=0$ $(0,0)$

Vertical asymptotes $(x+2)(x-1)=0$
 $x=-2$ or $x=1$

Approach

x	\rightarrow	-2	\rightarrow
$y = \frac{x^2}{(x+2)(x-1)}$	$\frac{(+)}{(+)(-)} \infty \frac{(+)}{(+)(-)}$		
	+	∞	-

x	\rightarrow	1	\rightarrow
$y = \frac{x^2}{(x+2)(x-1)}$	$\frac{(+)}{(+)(-)} \infty \frac{(+)}{(+)(+)}$		
	-	∞	+

horizontal / oblique asymptotes.

$$\begin{aligned} y &= \frac{x^2}{x^2+x-2} \\ &= 1 + \frac{2-x}{x^2+x-2} \end{aligned}$$

Divede out

$$\frac{x^2+x-2}{x^2+x-2} \frac{1}{-x+2}$$

As $x \rightarrow \pm\infty$, $y \rightarrow 1$ so $y=1$ is a horizontal asymptote

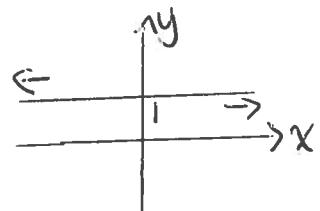
approach:

$$x \rightarrow +\infty$$

$y \rightarrow 1$ - a little bit
(below)

$$x \rightarrow -\infty$$

$y \rightarrow 1$ + a little bit
(above)

stationary points

$$y = 1 + \frac{2-x}{x^2+x-2} \leftarrow \text{use gradient rule}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(x^2+x-2)(-1) - (2-x)(2x+1)}{(x^2+x-2)^2} \\
 &= \frac{-x^2-x+2 - (4x-2x^2+2-x)}{(x^2+x-2)^2} \\
 &= \frac{x^2-4x}{(x^2+x-2)^2} \\
 &= \frac{x(x-4)}{(x^2+x-2)^2}
 \end{aligned}$$

For stationary points $\frac{dy}{dx} = 0$

$$x(x-4) = 0$$

$$x=0 \text{ or } x=4.$$

When $x=0$, $y=0$	$(0,0)$
$x=4$ $y = \frac{16}{15} = \frac{8}{9}$	$(4, \frac{8}{9})$

Nature

$\frac{dy}{dx} \rightarrow$ quite complicated

Easier to use nature tables

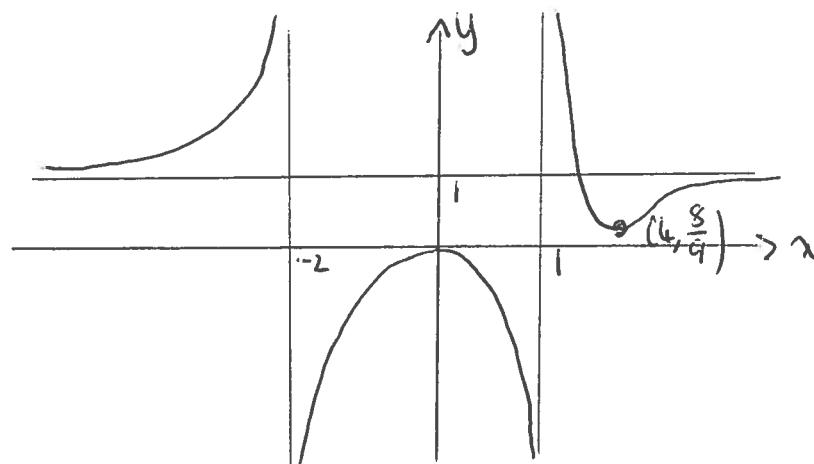
* * must use separate tables
because asymptotes mean
curve is not continuous.

x	\rightarrow	0	\rightarrow
$\frac{dy}{dx} = \frac{x(x-4)}{(x-1)^2}$		$\frac{(-)(-)}{(+)} 0 \quad \frac{(+)(-)}{(+)}$	
		+	-
shape .		/	\

Max TP $(0,0)$

x	\rightarrow	4	\rightarrow
$\frac{dy}{dx} = \frac{x(x-4)}{(x-1)^2}$		$\frac{(+)(-)}{(+)} 0 \quad \frac{(+)(+)}{(+)}$	
		-	+
		\	/

Min TP $(4, \frac{8}{9})$



Tips for success

- Differentiate from divided diff form \rightarrow (much easier)
- If using nature tables do them separately
- Remember graph can't cut vertical asymptotes.
(splits graph into slices)
but it can ~~can't~~ cut horizontal asymptotes.

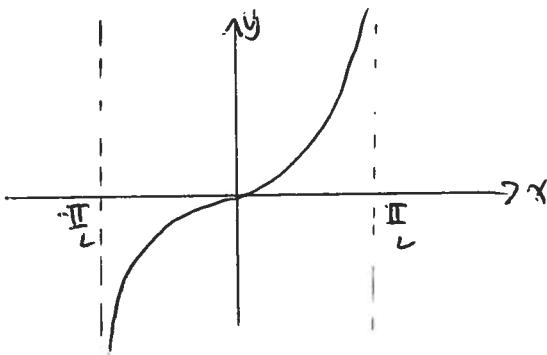
Maths in Action Unit 1 p112 Exercise 11 Questions 1 (a) , (d) , (f) , (g) , (h) ,
(k) , 3

Cfe p77 ex 5.11

Continuity/Discontinuity

Not all functions are differentiable for all x . If a function itself is not continuous, clearly it will not be differentiable where discontinuities occur.

- (a) Consider the graph and function $y = \tan x$. Where is $y = \tan x$ not differentiable in the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$? Why? Is $y = \tan x$ differentiable over an interval? If so, what is this interval?

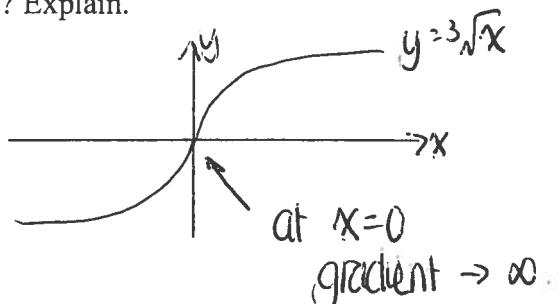


Not differentiable when
 $x = -\frac{\pi}{2}, \frac{\pi}{2}$.
 (since not defined here)
 $y = \tan x$ is differentiable
 for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

- (b) Consider the graph and function $y = \sqrt[3]{x}$. What happens to the gradient of the tangent to the curve as $x \rightarrow 0$? Is $y = \sqrt[3]{x}$ differentiable at $x=0$? Explain.

$$\begin{aligned} y &= x^{\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}} \end{aligned}$$

↑ undefined when $x=0$
 so $y = \sqrt[3]{x}$ is not differentiable
 at $x=0$.



Differentiability of a Function

This can be understood in terms of tangent lines approaching a point from the left and right.

As tangent lines approach from the left and right, the gradients of these tangent lines should approach the same value. The function should be ‘smooth’ and ‘curvy’ at the point in question.

Theorem:

A function is differentiable at a point if and only if both the left-hand and right-hand derivatives exist at that point and are equal to each other.

Thus, to determine if a function is not differentiable at a point, it suffices to show one of the following

- (i) The left hand derivative does not exist: $\nexists f'(a)$
- (ii) The right hand derivative does not exist: $\nexists f_+'(a)$
- (iii) If they exist, they are not equal : $f'(a) \neq f_+'(a)$

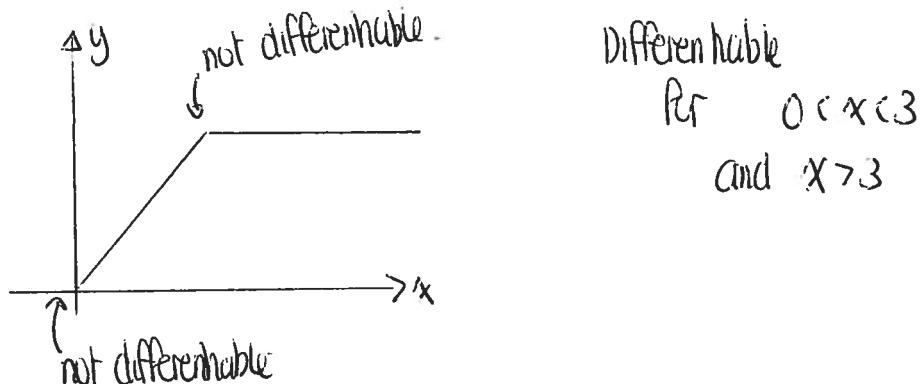
In terms of the graph of a function, this usually means identifying a sharp corner (aka ‘kink’) or an endpoint (at an endpoint obviously one of the derivatives won’t exist).

- (c) Determine where the function f defined by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 3, & x > 3 \end{cases}$$

Is differentiable and where it is not differentiable.

First sketch the graph of f .



□

□

Extrema of a Function

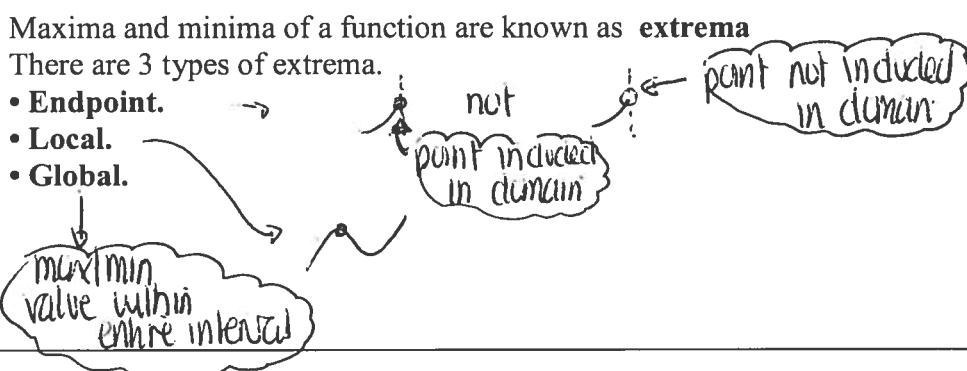
It is important to be able to identify the maximum and minimum values of a function.

Definition:

On any given function,
Critical points occur when either $f'(x) = 0$ or $f'(x)$ does not exist.

Definition:

Maxima and minima of a function are known as **extrema**.
There are 3 types of extrema.
• Endpoint.
• Local.
• Global.



Theorem:

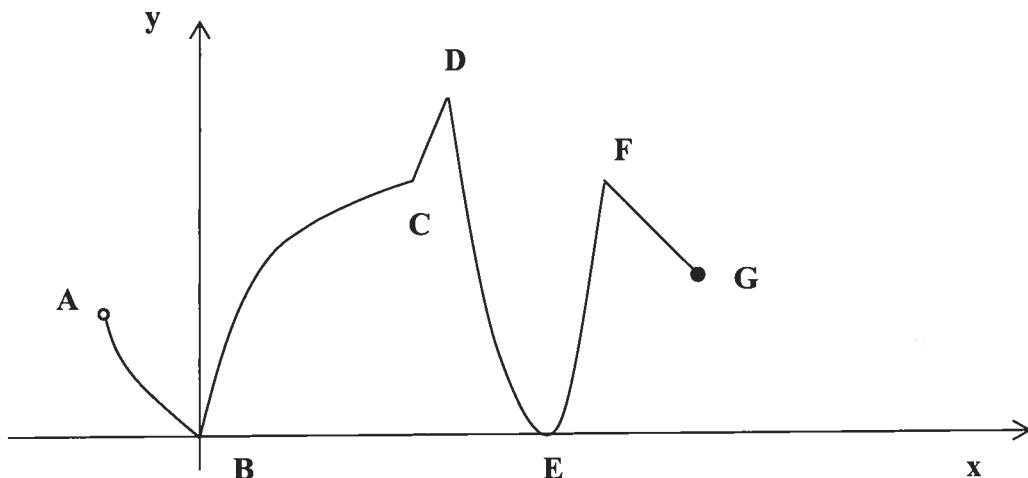
All local extrema occur at critical points.

Theorem:

Every global extremum is either a local extremum or endpoint extremum.

Example

Classify the extrema for the function f with the graph shown.



The small circles indicate the extent of where f is defined. The white circle means that the point is not in $\text{dom } f$, whereas the black circle means that the point is in $\text{dom } f$

Let's analyse the labelled points in turn. Remember, all extrema occur at critical points.

A is not in domain. \Rightarrow not a critical point

B is a global minimum point. ($f'(x)$ does not exist)

C is a critical point ($f'(x)$ does not exist). It is not a local extremum

D is a critical point since $f'(x)$ does not exist. It is a global maximum

E is a critical point $f'(x)=0$, It is a global minimum .

F is a critical point $\rightarrow f'(x)$ does not exist. It is a local minimum

G is a critical point $\rightarrow f'(x)$ does not exist. It is an endpoint minimum .

Stationary Points and Their Nature

Find the stationary points on each of the following curves and determine their nature:

1 $y = x^2 - 6x$

2 $f(x) = 9 - x^2$

3 $y = (5 - x)^2$

4 $y = 3x - x^3$

5 $f(x) = x(x - 3)^2$

6 $y = x^3 - 12x + 3$

7 $y = x^3(x - 4)$

8 $f(x) = 2x^3 - 3x^2 - x + 5$

9 $y = x^3(2 - x)$

1 (3, -9) min. TP

2 (0, 9) max. TP

3 (5, 0) min. TP

4 (-1, -2) min. TP; (1, 2) max. TP

5 (1, 4) max. TP; (3, 0) min. TP

6 (-2, 19) max. TP; (2, -13) min. TP

7 (0, 0) falling point of inflection; (3, -27) min. TP

8 (1.15, 2.92) min. TP; (-0.15, 5.08) max. TP

9 (0, 0) rising point of inflection; (1.5, 1.69) max. TP