AH DIFFERENTIATON 3

1 (a) $xy + y^2 = 2$

$$y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{(x+2y)}$$

(b) At $(1,1)$

$$\frac{dy}{dx} = -\frac{1}{1+2}$$

$$= -\frac{1}{3}$$

For equation of tangent

$$y - b = m(x-a)$$

$$y - 1 = -\tfrac{1}{3}(x-1)$$

$$3y - 3 = -x + 1$$

$$x + 3y - 4 = 0$$

2 $2y^2 - 2xy - 4y + x^2 = 0$

$$4y \cdot \frac{dy}{dx} - \left(2y + 2x \cdot \frac{dy}{dx}\right) - 4\frac{dy}{dx} + 2x = 0$$

$$(4y - 2x - 4)\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} = \frac{2(y-x)}{2(2y-x-2)}$$

$$\frac{dy}{dx} = \frac{y-x}{2y-x-2}$$

horizontal tangent $\Rightarrow \frac{dy}{dx} = 0$

$$\frac{y-x}{2y-x-2} = 0 \quad , \quad 2y - x \neq 2$$

$$y - x = 0$$

$$y = x \quad , \quad x \neq 2.$$

Given $\cancel{y^3+3xy}$

$$2y^2-2xy-4y+x^2=0 \quad , \quad y=x$$

$$\Rightarrow \quad 2x^2-2x^2-4x+x^2=0$$

$$x(x-4)=0$$

$$\underline{\underline{x=0\ ,\ 4}}$$

(3)

$$y^3+3xy=3x^2-5$$

$$3y^2\frac{dy}{dx}+3y+3x\frac{dy}{dx}=6x$$

$$(3y^2+3x)\frac{dy}{dx}=6x-3y$$

$$\frac{dy}{dx}=\frac{3(2x-y)}{3(y^2+x)}$$

$$\frac{dy}{dx}=\frac{2x-y}{x+y^2}$$

At $(2,1)$

$$\frac{dy}{dx}=\frac{2(2)-1}{2+1^2}$$

$$=\frac{3}{3}$$

$$=1$$

For equation

$$y-b=m(x-a)$$

$$y-1=x-2$$

$$\underline{\underline{y=x-1}}$$

(4) (a) $f(x)=x(1+x)^{10}$

$$f'(x)=(1+x)^{10}+10x(1+x)^9$$

$$=(1+x)^9\left[(1+x)+10x\right]$$

$$=\underline{\underline{(1+x)^9(1+11x)}}$$

4) (b) $y = 3^x$

$$\ln y = \ln 3^x$$

$$\ln y = x(\ln 3)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 3$$

$$\underline{\underline{\frac{dy}{dx} = 3^x(\ln 3)}}$$

5) (a) $f(x) = x^{\frac{1}{2}} e^{-x}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} e^{-x} + x^{\frac{1}{2}} \cdot (-e^{-x})$$

$$= x^{-\frac{1}{2}} e^{-x}\left(\frac{1}{2} - x\right)$$

$$= \frac{\frac{1}{2} - x}{\sqrt{x}\, e^x}$$

$$\underline{\underline{= \frac{1 - 2x}{2e^x \sqrt{x}}}}$$

(b) $y = \dfrac{(x+1)^2}{(x+2)^4}$

$$\ln y = \ln(x+1)^2 - \ln(x+2)^4$$

$$\ln y = 2\ln(x+1) - 4\ln(x+2)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{x+1} - \frac{4}{x+2}$$

~~(assuming x is a constant)~~

$$\frac{dy}{dx} = \cancel{\frac{4}{x+2}}\left(\frac{2}{x+1} - \frac{4}{x+2}\right) y$$

$$= \left(\frac{2}{x+1} + \frac{-4}{x+2}\right) y$$

Hence $\underline{\underline{a = 2, \quad b = -4}}$

6) $x = 5\cos\theta$ $y = 5\sin\theta$

(a) $\frac{dx}{d\theta} = -5\sin\theta$ $\frac{dy}{d\theta} = 5\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{5\cos\theta}{-5\sin\theta}$$

$$= \underline{\underline{-\cot\theta}}$$

(b) When $\theta = \frac{\pi}{4}$, $x = 5\cos\left(\frac{\pi}{4}\right)$ $y = 5\sin\left(\frac{\pi}{4}\right)$

$= 5 \cdot \frac{1}{\sqrt{2}}$ $= 5 \cdot \frac{1}{\sqrt{2}}$

$= \frac{5\sqrt{2}}{2}$ $= \frac{5\sqrt{2}}{2}$

Given $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$

and $m = -1$

For equation, $y - b = m(x - a)$

$$y - \frac{5\sqrt{2}}{2} = -\left(x - \frac{5\sqrt{2}}{2}\right)$$

$$2y - 5\sqrt{2} = -2x + 5\sqrt{2}$$

$$2y = -2x + 10\sqrt{2}$$

$$\underline{\underline{y = -x + 5\sqrt{2}}}$$

7) $x = t^2 + t - 1$ $y = 2t^2 - t + 2$ $A(\overset{x}{-1}, \overset{y}{5})$

$-1 = t^2 + t - 1$

$0 = t^2 + t$

$0 = t(t+1)$

$t = 0$ or $t = -1$

When $t = 0 \Rightarrow y = 2$

When $t = -1 \Rightarrow y = 2(-1)^2 - (-1) + 2$

$= 2 + 1 + 2$

$= 5$

Hence $A(-1, 5)$ is on the curve, when $t = -1$

(7) Continued...

$$\frac{dx}{dt} = 2t + 1 \qquad \frac{dy}{dt} = 4t - 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{4t-1}{2t+1}, \quad \text{At } t = -1$$

$$= \frac{4(-1)-1}{2(-1)+1}$$

$$= \frac{-5}{-1}$$

$$= 5$$

For $A(-1, 5)$, $m_{tgt} = 5 \Rightarrow y - b = m(x - a)$

$$y - 5 = 5(x + 1)$$

$$\underline{\underline{y = 5x + 10}}$$

(8) $s(t) = \frac{1}{3}t^3 + t^2 - 8t + 10$

(a) $s(0) = 10$

$\Rightarrow$ the object was 10 m from the origin.

(b) stationary when $v = 0$

$s'(t) = t^2 + 2t - 8$

Now $t^2 + 2t - 8 = 0$

$(t+4)(t-2) = 0$

$t = \cancel{-4}, 2$

N/A

$\Rightarrow$ the object was stationary at $t = 2$ secs.

(c) $s'(t) = t^2 + 2t - 8$

$s'(5) = 5^2 + 2(5) - 8$

$= 35 - 8$

$= 27 \ ms^{-1}$

$s''(t) = 2t + 2$

$s''(5) = 2(5) + 2$

$= 12 \ ms^{-2}$

At $t = 5$ secs, the object is travelling forwards at $27 \ ms^{-1}$ and accelerating at $12 ms^{-2}$

(d) When does $a \leq 0$?

$s''(t) \leq 0$

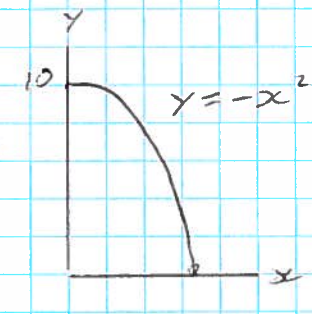
$2t + 2 \leq 0$

$2t \leq -2$

$t \leq -1$

Since $t \leq -1$ will not occur the object never decelerates or has a constant velocity

(9)



$x = \frac{1}{2}t$ $\quad y = -\left(\frac{1}{2}t\right)^2 + 10$

$= -\frac{1}{4}t^2 + 10$

Now $\frac{dx}{dt} = \frac{1}{2}$, $\frac{dy}{dt} = -\frac{1}{2}t$

$|v| = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}t\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{1}{4}t^2}$

The berry hits the ground when $y = 0$

$\Rightarrow -\frac{1}{4}t^2 + 10 = 0$

$10 = \frac{1}{4}t^2$

$40 = t^2$

$t = \sqrt{40}$

$= 2\sqrt{10}$ Secs

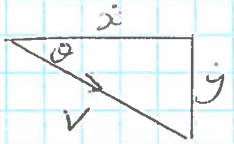
When $t = 2\sqrt{10}$,

$|v| = \sqrt{\frac{1}{4}(1 + t^2)}$

$= \frac{1}{2}\sqrt{1 + t^2}$

$= \frac{1}{2}\sqrt{1 + (2\sqrt{10})^2} = \frac{1}{2}\sqrt{1 + 40} = \underline{\underline{\frac{1}{2}\sqrt{41} \ ms^{-1}}}$

(9) Continued...



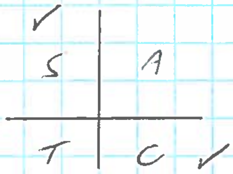
$\dot{x} = \frac{1}{2}$ $\qquad$ $\dot{y} = -\frac{1}{2}t$

$\dot{x}(2\sqrt{10}) = \frac{1}{2}$ $\qquad$ $\dot{y}(2\sqrt{10}) = -\sqrt{10}$

Now $\tan\theta = \dfrac{\dot{y}}{\dot{x}}$

$= \dfrac{-\sqrt{10}}{\frac{1}{2}}$

$= -2\sqrt{10}$



r.a. = 81.0°

Since $\dot{x} > 0$ and $\dot{y} < 0$

$\Rightarrow$ 4th quadrant

Hence direction of velocity = $\underline{\underline{-81.0^\circ}}$

(10) $y = \dfrac{\sin x}{x^2}$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{x^2\cos x - 2x\sin x}{x^4}$$

$$= \frac{x(x\cos x - 2\sin x)}{x^4} = \frac{x\cos x - 2\sin x}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\cos x}{x^2} - \frac{2\sin x}{x^3}\right)$$

$$= \left(\frac{-x^2\sin x - 2x\cos x}{x^4}\right) - \left(\frac{2x^3\cos x - 6x^2\sin x}{x^6}\right)$$

$$= \frac{-x^2\sin x - 2x\cos x}{x^4} - \left(\frac{2x^2(x\cos x - 3\sin x)}{x^6}\right)$$

$$= \frac{-x^2\sin x - 2x\cos x}{x^4} - \frac{2(x\cos x - 3\sin x)}{x^4}$$

$$= \frac{-x^2\sin x - 2x\cos x - 2x\cos x + 6\sin x}{x^4}$$

$$= \frac{-x^2\sin x - 4x\cos x + 6\sin x}{x^4}$$

Now $x^2 \cdot \frac{d^2y}{dx^2} + 4x \cdot \frac{dy}{dx} + (x^2+2)y$

$$= x^2\left(\frac{-x^2\sin x - 4x\cos x + 6\sin x}{x^4}\right) + 4x\left(\frac{x\cos x - 2\sin x}{x^3}\right)$$

$$+ (x^2+2)\left(\frac{\sin x}{x^2}\right)$$

$$= \frac{-x^2\sin x - 4x\cos x + 6\sin x}{x^2} + \frac{4(x\cos x - 2\sin x)}{x^2}$$

$$+ \frac{x^2\sin x + 2\sin x}{x^2}$$

$$= \frac{-x^2\sin x + x^2\sin x - 4x\cos x + 4x\cos x + 6\sin x - 8\sin x + 2\sin x}{x^2}$$

$$= \frac{0}{x^2}$$

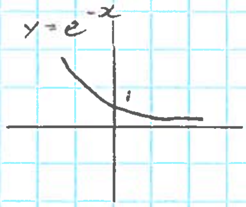
$$= 0 \quad , \quad x \neq 0.$$

(11) $y = x^3 e^{-x}$

$$\frac{dy}{dx} = 3x^2 e^{-x} + x^3\left(-e^{-x}\right)$$
$$= x^2\left(3e^{-x} - xe^{-x}\right)$$
$$= x^2 e^{-x}(3-x)$$

$$\frac{d^2y}{dx^2} = 2xe^{-x}(3-x) + \left(-e^{-x}\right)x^2(3-x) + (-1)x^2e^{-x}$$
$$= x\left[2e^{-x}(3-x) - xe^{-x}(3-x) - xe^{-x}\right]$$

Stat pts when $\frac{dy}{dx} = 0$



$$x^2e^{-x}(3-x) = 0$$

$x^2 = 0$ or $e^{-x} = 0$ or $3 - x = 0$

$x = 0$ No solution $x = 3$

when $x = 0$, $y = 0$, $\frac{d^2y}{dx^2} = 0$ $\Rightarrow$ Use nature table

x	-1	0	1
$f'(x)$	$+$	0	$+$
shape	/	—	/

$f'(-1) = (-1)^2 \cdot e^1 \cdot (3+1) > 0$

$f'(1) = 1^2 \cdot e^{-1}(3-1) > 0$

$\Rightarrow$ There is a rising pt of inflexion at $(0,0)$

when $x = 3$, $y = \frac{27}{e^3}$, $\frac{d^2y}{dx^2} = 3\left[2e^{-3}\times 0 - 3e^{-3}\times 0 - 3e^{-3}\right]$
$$= -9e^{-3} \quad (<0)$$

$\Rightarrow$ There is a max TP at $\left(3, \frac{27}{e^3}\right)$