# DIFFERENTIAL EQUATIONS

## Solution of First Order Differential Equations Differential Equations

A differential equation is an equation containing a derivative (or differential quotient).

e.g. $y\dfrac{dy}{dx} + 2xy = x$ is a **first order** differential equation

$\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4 = \sin x$ is a **second order** differential equation

**To solve differential equations :-**

- **we find a function $y$ which satisfies the original equation**
- **we use integration**

Differential equations may have a **general solution :-**

e.g. $\dfrac{dy}{dx} = 2x + 3$

$$\Rightarrow \int dy = \int (2x+3)dx$$

$$\Rightarrow y = x^2 + 3x + C$$

or the same differential equation may have a **particular solution**, if **initial conditions** are given :-

e.g. Solve $\dfrac{dy}{dx} = 2x + 3$ , given that $y$=6 when $x$=1 integrate $y = \int (2x+3)\,dx$

$y = x^2 + 3x + c$

When $x = 1$, $y = 6$ so $6 = 1 + 3 + c$

$c = 2$

Solution $y = x^2 + 3x + 2$

**Maths in Action Book 2 p74 Exercise 6 Questions 1(a) , (d) , (e) , (f) , (j) , (k)**
**2(a) , (c) , (e) 3 , 4 , 5 , 6**

# Type 1 - Variables Separable

If the parts of the differential equation involving x and y can be separated from each other to give

$$\frac{dy}{dx} = f(x)g(y)$$

then we can rearrange to give

all y's on this side

$$\frac{dy}{g(y)} = f(x)dx$$

all x's on this side

and integrate both sides.

**Examples**

1. Solve $(x-1)\frac{dy}{dx} = y$

Re arrange $\frac{dy}{y} = \frac{dx}{x-1}$

Integrate $\int \frac{dy}{y} = \int \frac{dx}{x-1}$

$\ln y = \ln|x-1| + c$

** To get rid of logs we can let $c = \ln k$

$\ln y = \ln|x-1| + \ln k$

$\ln y = \ln k|x-1|$

$y = k|x-1|$

*** Always get answer in the form $y = \ldots$

**2.** Find the general solution of $\frac{dy}{dx} = e^{2x-3y}$ giving y in terms of x.

$\frac{dy}{dx} = e^{2x}e^{-3y}$

$e^{3y}\,dy = e^{2x}\,dx$

Integrate $\int e^{3y}\,dy = \int e^{2x}\,dx$

$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2x} + c$

$e^{3y} = \frac{3}{2}e^{2x} + d$

Take logs
$\ln e^{3y} = \ln\left(\frac{3}{2}e^{2x} + d\right)$

$3y = \ln\left(\frac{3}{2}e^{2x} + d\right)$

$y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2x} + d\right)$

p77 Ex8
①

**3.** Find the particular solution of $x^2\frac{dy}{dx} = 1$, given that $y=3$ when $x=1$.

$x^2\frac{dy}{dx} = 1$

$dy = \frac{1}{x^2}dx$

Integrate $\int dy = \int x^{-2}\,dx$

$y = \frac{x^{-1}}{-1} + c$

$y = -\frac{1}{x} + c$

When $x=1$, $y=3$

$3 = -1 + c$

$c = 4$

So $y = 4 - \frac{1}{x}$

**4.** Find the particular solution of $\frac{dy}{dx} = xy - 2x$, given that $y=5$ when $x=0$.

$$\frac{dy}{dx} = xy - 2x$$

Factorise first $\quad \frac{dy}{dx} = x(y-2)$

$$\frac{dy}{y-2} = x\,dx$$

$$\int \frac{dy}{y-2} = \int x\,dx$$

$$\ln|y-2| = \frac{x^2}{2} + c.$$

When $x=0$, $y=5$

$$\ln 3 = c.$$

Usually easier to find c before rearranging to find y

so $\ln|y-2| = \frac{x^2}{2} + \ln 3$

$$\ln \frac{|y-2|}{3} = \frac{x^2}{2}$$

$$\frac{y-2}{3} = e^{\frac{x^2}{2}}$$

$$y = 3e^{\frac{x^2}{2}} + 2$$

**5.** Find the particular solution of $(1+x^2)\frac{dy}{dx} = 4e^y$, given that $y=0$ when $x=0$.

$$(1+x^2)\frac{dy}{dx} = 4e^y$$

$$\frac{dy}{4e^y} = \frac{1}{1+x^2}\,dx$$

Integrate $\int \frac{dy}{4e^y} = \int \frac{1}{1+x^2}\,dx$

$$-\frac{1}{4}e^{-y} = \tan^{-1}x + c$$

When $x=0, y=0$

$$-\frac{1}{4} = \tan^{-1}0 + c$$

$$c = -\frac{1}{4}$$

$$-\frac{1}{4}e^{-y} = \tan^{-1}x - \frac{1}{4}$$

$$e^{-y} = 1 - 4\tan^{-1}x$$

$$-y = \ln|1 - 4\tan^{-1}x|$$

$$y = -\ln|1 - 4\tan^{-1}x|$$

***Tips for success.....***

- If solving a first order d.e ie only contains $\frac{dy}{dx}$ (not $\frac{d^2y}{dx^2}$) try separating variables so y's on one side and x's on other.
- Integrate + c.
- Final answer must be written as y = **
- If given initial conditions find c first before re arranging.

**Maths in Action Book 2** ~~**page 74; Exercise 6; Q1c, f, g; Q2c, e; Q4.**~~
**page 76; Exercise 7; Q1c, d, h; Q2a, b, e.**
✱ **page 77; Exercise 8; Q1a, c, e, g, h; Q2, Q3, Q4, Q5, Q6, Q7, Q10.**

# Type 2 Linear First Order Differential Equations

Linear first order differential equations are those that can be written in the form

x's  x's only

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are expressions in x.

$\frac{dy}{dx}$ + term with plain y

**How to Solve**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We multiply through by $e^R$ where $R = \int P(x)dx$

what is infront of y

This gives $$e^R \frac{dy}{dx} + e^R P(x)y = e^R Q(x)$$

The left hand side is now an exact derivative.

$$\frac{d}{dx}(e^R y) = e^R Q(x)$$

$$\frac{d}{dx}(e^R y) = e^R \frac{dy}{dx} + ye^R \frac{dR}{dx} = e^R \frac{dy}{dx} + e^R P(x).y$$

since $R = \int P(x)dx$, $\frac{dR}{dx} = P(x)$

Integrate $$e^R y = \int e^R Q(x)dx + c$$

Note $e^R$ converts the whole of the left side into the derivative of the one function and it is called the integrating factor.

Learn

Method 1. Put the equation into the form $\frac{dy}{dx} + P(x)y = Q(x)$

2. Multiply through by $e^{\int P(x)dx}$

3. Put LHS = $\frac{d}{dx}(e^R y)$ integrating factor. 4. Integrate

Examples

1. Solve $\frac{dy}{dx} + \frac{y}{x} = 1$

$$\frac{dy}{dx} + \frac{1}{x}y = 1$$

Linear since $\frac{dy}{dx} + \dots y = \dots$

Integrating factor

$$e^{\int \frac{1}{x}dx}$$

whats infront of y

$$= e^{\ln x}$$

$$= x$$

VERY IMPORTANT - don't put constant of integration in here. It is taken care of later

Multiply by x

$$x\frac{dy}{dx} + y = x$$

becomes

$$\frac{d}{dx}(xy) = x$$

integrating factor xy

Integrate both sides

$$xy = \int x\,dx$$

$$xy = \frac{x^2}{2} + c$$

remember +c

$$y = \frac{x}{2} + \frac{c}{x}$$

rearrange to get y

2. Solve $x\frac{dy}{dx} - (x-2)y = \frac{1}{x}$

* Make plain $\frac{dy}{dx}$ first *

$$\frac{dy}{dx} - \frac{x-2}{x} y = \frac{1}{x^2}$$ linear

Integrating factor

$$e^{\int -\frac{x-2}{x} dx}$$

$$= e^{\int \left(\frac{2}{x} - 1\right) dx}$$

* Note take negative with term in front of y

$$= e^{2\ln x - x}$$

$$= \frac{e^{2\ln x}}{e^x}$$

$$= \frac{e^{\ln x^2}}{e^x}$$

$$= \frac{x^2}{e^x}$$

Multiply by integrating factor

$$\frac{x^2}{e^x}\frac{dy}{dx} - \frac{x^2(x-2)}{e^x x} y = \frac{1}{e^x}$$

becomes

$$\frac{d}{dx}\left(\frac{x^2}{e^x} y\right) = \frac{1}{e^x}$$

integrating factor $\times y$

Integrate both sides

$$\frac{x^2}{e^x} y = \int \frac{1}{e^x} dx$$

$$\frac{x^2}{e^x} y = -e^{-x} + c$$

$$\frac{x^2}{e^x} y = c - \frac{1}{e^x}$$

$$y = \frac{ce^x}{x^2} - \frac{1}{x^2}$$

p114 Ex1 Q3

3. Find the particular solution of

$$x(x+1)\frac{dy}{dx} + y = (x+1)^2 e^x \quad \text{given that x = 1 when y = 0}$$

Make plain $\frac{dy}{dx}$

$$\frac{dy}{dx} + \frac{1}{x(x+1)}y = \frac{x+1}{x}e^x \qquad (*)$$

Integrating factor $e^{\int \frac{1}{x(x+1)}dx}$

partial fractions

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

let $x=0$ $A=1$

$x=-1$ $B=-1$

So $e^{\int \frac{1}{x} - \frac{1}{x+1}dx}$

$= e^{\ln x - \ln(x+1)}$

$= e^{\ln \frac{x}{x+1}}$

$= \frac{x}{x+1}$

Multiply $(*)$ by $\frac{x}{x+1}$

$$\frac{x}{x+1}\frac{dy}{dx} + \frac{1}{(x+1)^2}y = e^x$$

$$\frac{d}{dx}\left(\frac{x}{x+1}y\right) = e^x$$

Integrate $\frac{x}{x+1}y = e^x + C$

When $y=0$ $x=1$, $0 = e + C \Rightarrow C = -e$

so $\frac{x}{x+1}y = e^x - e$

$$y = \frac{x+1}{x}(e^x - e)$$

***Tips for success.....***

$\frac{dy}{dx} + \_\_ y = \_\_ \Rightarrow$ linear type.

- make sure plain $\frac{dy}{dx}$
- Find integrating factor $e^{\int (\text{bit in front of } y)\, dx}$
- multiply by integrating factor so LHS $= \frac{d}{dx}(\text{integrating factor} \times y)$
- Integrate
- Find c if appropriate
- write answer as $y =$

**Maths in Action Book 3 Page 114 Exercise Question 3**

**Page 116 Exercise 2 Questions 1 (a),(b),(d) , (e) Q2, Q3,**

# Applications of First Order Differential Equations

A large number of situations where *rates of change* are involved can be modelled using differential equations.

Common applications include :-

- Populations: growth and decay
- Law of cooling: where the rate at which an object cools is proportional to the temperature difference between it and its surroundings.

Given the situation we must first construct a model (in the form of a differential equation) and the use the usual procedures to solve.

Note

1. Remember $\frac{dy}{dx}$ means rate of change of $y$ with $x$.
2. Rate of change of P with t varies directly as t means

$$\frac{dP}{dt} \propto t \quad \text{ie} \quad \frac{dP}{dt} = kt \qquad (k = \text{constant})$$

3. Rate of change of Q with r varies inversely as r means

$$\frac{dQ}{dr} \propto \frac{1}{r} \quad \text{ie} \quad \frac{dQ}{dr} = \frac{k}{r}$$

4. For motion of a body in a straight line

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt}$$

**Example 1**

The rate at which a radioactive material decays at any instant is proportional to the mass remaining at any instant. Given that there are $x$ grams present after $t$ days, the differential equation in $x$ is modelled by :-

$$\frac{dx}{dt} = -kx$$

The half-life (i.e. the time taken for half the mass to decay) of the radioactive material is 25 days.

Find the time taken for 100g of the material to decay to 20g.

$$\frac{dx}{dt} = -kx$$

$$\frac{dx}{x} = -k\,dt$$

Integrate $\displaystyle\int \frac{dx}{x} = \int -k\,dt$

$$\ln x = -kt + c$$

When $t = 0$, $x = 100 \Rightarrow \ln 100 = c$

$t = 25$ $x = 50 \Rightarrow \ln 50 = -25k + \ln 100$

$$\ln \tfrac{1}{2} = -25k$$

$$k = -\frac{1}{25}\ln\frac{1}{2}$$

So $\ln x = \left(\frac{1}{25}\ln\frac{1}{2}\right)t + \ln 100$

When $x = 20$ $\ln 20 = \left(\frac{1}{25}\ln\frac{1}{2}\right)t + \ln 100$

$$\ln\frac{1}{5} = \left(\frac{1}{25}\ln\frac{1}{2}\right)t$$

$$t = \frac{25\ln\frac{1}{5}}{\ln\frac{1}{2}}$$

$$= 58 \text{ days.}$$

**Example 2**

An infectious disease spreads at a rate which is proportional to the product of the number infected and the number uninfected.

Initially one half of the population is infected and the rate of spread is found to be $\frac{1}{48}$ of the population per day.
Calculate the proportion of the population which will be infected after 12 days.

let $P$ = number of population (constant)
$x$ = number of people infected

so $P-x$ = number uninfected

We have $\frac{dx}{dt} = kx(P-x)$ ← proportional to product of number infected and uninfected

Initially $t=0$ $\frac{dx}{dt} = \frac{P}{48}$ and $x = \frac{P}{2}$

$$\frac{P}{48} = k\frac{P}{2}\left(\frac{P}{2}\right)$$

$$P = 12kP^2$$

$$k = \frac{1}{12P} \quad \text{(since } P \neq 0\text{)}$$

So $\frac{dx}{dt} = \frac{1}{12P}x(P-x)$

$\Rightarrow \int \frac{P}{x(P-x)}\,dx = \int \frac{1}{12}\,dt$

↑ partial fractions → $\frac{P}{x(P-x)} = \frac{A}{x} + \frac{B}{P-x}$

$P = A(P-x) + Bx$

When $x=0$ $A=1$
$x=P$ $B=1$

so $\int \frac{1}{x} - \frac{1}{P-x}\,dx = \int \frac{1}{12}\,dt$

$$\ln x - \ln(P-x) = \frac{1}{12}t + c$$

$$\ln \frac{x}{P-x} = \frac{1}{12}t + c$$

When $t=0$ $x = \frac{1}{2}P$

$$\ln\left(\frac{\frac{1}{2}P}{\frac{1}{2}P}\right) = c$$

$$c = 0$$

So $\ln \frac{x}{P-x} = \frac{1}{12}t$

$$\frac{x}{P-x} = e^{\frac{t}{12}}$$

When $t=12$

$$\frac{x}{P-x} = e$$

$$x = Pe - xe$$

$$x(1+e) = Pe$$

proportion infected

$$\frac{x}{P} = \frac{e}{1+e}$$

$$= 0.731 \quad (3 s.f)$$

$$= 73.1\%$$

**Maths in Action Book 2 page 81; Exercise 9A; Q1, 3, 4, 5, 6, 7, 8, 10.**
**page 83; Exercise 9B; Q1, 2, 3, 5, 7, 12.**

***Tips for success.....***

- Remember to define any letters used if not given in question.
- varies directly ⇒ letter goes on top
- varies inversely ⇒ letter goes on bottom
- Solve as before.

# Solving Quadratic Equations when the Discriminant is Negative

We define $\sqrt{-1} = i$

eg $\sqrt{-4} = \sqrt{4} \times \sqrt{-1}$ $= 2i$ $\qquad$ $\sqrt{-81} = \sqrt{81} \times \sqrt{-1}$ $= 9i$

These are called imaginary numbers.

Complex numbers have a real part and an imaginary part. eg 2 + 5i , 6 - 8i

The set of complex numbers is denoted by C

All quadratic equations can be solved if we use complex numbers.

Example

Solve $x^2 + 2x + 5 = 0$ with $x \in C$

$a = 1 \qquad b = 2 \qquad c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$x = -1 \pm 2i$$

# Second Order Differential Equations

**"Simple" Type:-** **i.e.** $\frac{d^2y}{dx^2} = f(x)$

Solve these by integrating **twice.**

e.g. $$\frac{d^2y}{dx^2} = x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3}{3} + c$$

$$\Rightarrow y = \frac{x^4}{12} + cx + d$$

**Note that like all 2nd order differential equations the solution MUST have 2 arbitrary constants.**

# Second Order *Linear* Differential Equations

In general, these take the form :- $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q(x)$

$\frac{d^2y}{dx^2}$ must be there ie $a \neq 0$

but $\frac{dy}{dx}$ or $y$ could be missing

### (a) Homogeneous 2nd Order Linear Differential Equations

Initially we will concentrate on the case when $Q(x)=0$.

This is known as a **homogeneous 2nd order differential equation:-**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{.............. equation (1)}$$

The differential equation $b\frac{dy}{dx} + cy = 0$ has a solution of the form $y = Ae^{mx}$

$$y = Ae^{mx}$$

$$\Rightarrow \frac{dy}{dx} = Ame^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

$b\frac{dy}{dx} = -cy$

$\frac{dy}{y} = -\frac{c}{b}\,dx$

$\ln y = -\frac{c}{b}x + k$

$y = e^{-\frac{c}{b}x + k}$

$y = e^k e^{-\frac{c}{b}x}$

$y = Ae^{mx}$

Substituting $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the original equation (1), gives :-

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0 \,, \quad \text{dividing all by } Ae^{mx} \text{ (assuming } Ae^{mx} \neq 0\text{)}$$

we obtain $\quad am^2 + bm + c = 0$

This is known as the **Auxiliary Equation** and its solutions for $m$ provide 2 values of $m$ which make $y = Ae^{mx}$ a valid solution for our original equation (1).

Distinguishing between the 2 roots of $m$ using $m_1$ and $m_2$ give us 2 solutions for equation 1.

Conclusion $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ has auxilary equation $am^2+bm+c=0$

The roots of the Auxiliary Equation may take 3 forms :-

(i) **2 real distinct roots** $(b^2 - 4ac > 0)$

$m = m_1$ and $m_2$ and the general solution of equation (1) is :-

$$y = Ae^{m_1x} + Be^{m_2x}$$

(ii) **Equal roots** $(b^2 - 4ac = 0)$

$m = m_1$ (twice) and the general solution of equation (1) is :-

$$y = Ae^{m_1x} + Bxe^{m_1x} = (A + Bx)e^{m_1x}$$

see AH3 page 119

(iii) **2 complex roots** $(b^2 - 4ac < 0)$

$m = m_1$ and $m_2$, but $m_1$ and $m_2$ are of the form $p \pm qi$ and the general solution of equation (1) is :-

$$y = Ae^{px}\cos qx + Be^{px}\sin qx)$$

see AH3 page121

* * LEARN

**Example 1**

Find the general solution of :-

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Auxilary equation $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3$ or $m = 2$ (2 distinct roots)

General solution

$y = Ae^{-3x} + Be^{2x}$

**Example 2**

Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=0$

auxilary equation $m^2+2m+5=0$

doesn't factorise $\Rightarrow$ use quadratic formula

$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $a=1,\ b=2,\ c=5$

$m=\frac{-2\pm\sqrt{2^2-20}}{2}$

$\sqrt{-16}=\sqrt{16}\sqrt{-1}$
$=4i$
imaginary numbers
$\sqrt{-1}=i$

$m=\frac{-2\pm 4i}{2}$

$m=-1\pm 2i$

General solution

$y=Ae^{-x}\cos 2x+Be^{-x}\sin 2x$

**Maths in Action Book 3 page 119 Exercise 3 Question 1(a) (d) (e)**
**page 120 Exercise 4 Question 1(a) (d) (e)**
**page 122 Exercise 5 Question 1(a) (d) (e)**

**Example 3**

Given $y(0) = 2$ and $y'(0) = 9$, solve the differential equation:-

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

auxilary equation

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3$ equal roots.

General equation

$y = Ae^{3x} + Bxe^{3x}$ (*)

need $x$ here so this term is different from $Ae^{3x}$

$y(0) = 2$

ie $x=0, y=2$ so $2 = A$

$y'(0) = 9$

ie $x=0$ $\frac{dy}{dx} = 9$

Differentiate (*)

$\frac{dy}{dx} = 3Ae^{3x} + \underbrace{Bx.3e^{3x} + Be^{3x}}_{\text{product rule}}$.

When $x=0$ $\frac{dy}{dx} = 9$ so $9 = 3A + B$

$9 = 6 + B$

$B = 3$

Solution $y = 2e^{3x} + 3xe^{3x}$

**Maths in Action Book 3 page 119 Exercise 3 Question 2(a) (d)**
**page 120 Exercise 4 Question 2(a) (d)**
**page 122 Exercise 5A Question 1(a) (d)**
**page 122 Exercise 5B**

# General Solution of a Non-Homogeneous Equation

We are now going to consider solving second order differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad \text{....................( 1)}$$

**First** we find the general solution to the corresponding homogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad \text{.............................(2)}$$

The solution is the form

$$y = Ag(x) + Bh(x)$$

This is called the complementary function C. F.

**Second** we find a particular solution (particular integral) P.I. of the non-homogeneous equation (1)

$$y = P(x)$$

The general solution we want is

$$y = Ag(x) + Bh(x) + P(x)$$

or $\boxed{y = C.F + P.I.}$

<u>How to find the P.I.</u>
Look at f(x) the right hand side of (1) and choose the correct P.I. from the list.

| f(x) | Correct form for P.I. |
|---|---|
| $e^{px}$ | $y = Ce^{px}$ (p not a root of the auxiliary equation)<br>$y = Cxe^{px}$ ( p a non repeated root of the auxiliary equation)<br>$y = Cx^2e^{px}$ (p a repeated root of the auxiliary equation) |
| Hcospx + Ksinpx<br><br>(or either of these by itself) | y = Ccospx + Dsinpx but if this is already the form of the C.F. then use y = x(Ccospx + Dsinpx) |
| linear function<br>px + q | y = Cx + D (m = 0 not a root of the auxiliary equation)<br>y = x(Cx + D) ( m = 0 a non-repeated root) |
| quadratic function<br>$rx^2$ + px +q | $y = Cx^2 + Dx + E$ (m=0 not a root of the auxilary equation)<br>$y = x(Cx^2 + Dx + E)$ (m = 0 a non repeated root) |

**<u>Finally</u>** substitute your P.I. as chosen above in the original differential equation to find constants C and D.

Example

Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x + 4$

auxilary equation

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -2$ or $m = -1$

C.F $y = Ae^{-2x} + Be^{-x}$

P.I. Consider RHS $= 4x + 4$

Try $y = Cx + D$

$\frac{dy}{dx} = C$

$\frac{d^2y}{dx^2} = 0$

Substitute into $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x + 4$

$0 + 3C + 2Cx + 2D = 4x + 4$

Equate coefficients

constant $3C + 2D = 4$ --- ①

x term $2C = 4$

$C = 2$

In ① $6 + 2D = 4$

$2D = -2$

$D = -1$

P.I $y = 2x - 1$

General solution $y = CF + PI$

$y = Ae^{-2x} + Be^{-x} + 2x - 1$

**Example**

Solve $\qquad \dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 30\sin 2x$

auxilary equation

$$m^2 + 6m + 10 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 40}}{2}$$

$$m = \frac{-6 \pm 2i}{2}$$

$$m = -3 \pm i$$

CF $\quad y = Ae^{-3x}\cos x + Be^{-3x}\sin x$

PI $\quad$ look at $\;$ RHS $= 30\sin 2x$

Try $\quad y = C\cos 2x + D\sin 2x$

NB must have both trig bits.

$$\frac{dy}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2y}{dx^2} = -4\cos 2x - 4D\sin 2x$$

Substitute into $\quad \dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 30\sin 2x$

$$-4C\cos 2x + 4D\sin 2x - 12C\sin 2x + 12D\cos 2x + 10C\cos 2x$$
$$+ 10D\sin 2x = 30\sin 2x$$

Equate coefficients

$(\cos 2x)$ $\qquad -4C + 12D + 10C = 0$

$$12D + 6C = 0$$

$$C = -2D$$

$(\sin 2x)$ $\quad -4D - 12C + 10D = 30$

$-4D + 24D + 10D = 30$

$30D = 30$

$D = 1$

$C = -2$

P.I $\quad y = -2\cos 2x + \sin 2x$

General solution $\quad y = CF + PI$

$y = Ae^{-3x}\cos x + Be^{-3x}\sin x - 2\cos 2x + \sin 2x.$

Example

Find the particular solution to the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

given that x = 0, y = 1 and $\frac{dy}{dx} = 2$

auxilary equation $\quad m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2$ or $m = 1$

CF $\quad y = Ae^{2x} + Be^x$

PI $\quad$ Try $y = Cxe^x$ (since $e^x$ in CF already)

$\frac{dy}{dx} = Cxe^x + Ce^x$

$\frac{d^2y}{dx^2} = Cxe^x + Ce^x + Ce^x$

$= Cxe^x + 2Ce^x$

Substitute into $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = e^x$

$Cxe^x + 2Ce^x - 3(Cxe^x + Ce^x) + 2(Cxe^x) = e^x$

$Cxe^x + 2Ce^x - 3Cxe^x - 3Ce^x + 2Cxe^x = e^x$

$-Ce^x = e^x$

$C = -1$

P.I $y = -xe^x$

General solution $y = CF + PI$

$y = Ae^{2x} + Be^x - xe^x$ (*)

Now

When $x=0,\ y=1 \Rightarrow 1 = A+B \ \dots ①$

From (*) $\frac{dy}{dx} = 2Ae^{2x} + Be^x - (xe^x + e^x)$

When $x=0,\ \frac{dy}{dx} = 2$ so $2 = 2A + B - 1$

$2A+B = 3 \ \dots ②$

② - ① gives $A=2$

$\Rightarrow B=-1$

Particular solution $y = 2e^{2x} - e^x - xe^x$

**Maths in Action Book 3 Page 126 Exercise 7A Questions 1(a), (b) ,(e), (f), (g), (j)** 2
**Page 127 Exercise 7B Questions !(a) ,(e),**