ADVANCED HIGHER MATHEMATICS

COURSE ASSESSMENT UNIT 1

TIME ALLOWED 1 HOUR 30 MINUTES

1. Differentiate with respect to x

   (i) $e^{\cos x}$ (ii) $\dfrac{\sin x}{1+\cos^2 x}$ **2 , 4**

2. Expand

   $$(2x^2 - \frac{3}{x})^5 \qquad x \neq 0$$ **5**

   and simplify as far as possible.

3. Let f be the function defined by

   $$f(x) = \frac{3x-1}{x^2(x-3)}$$

   Show that

   $$f'(x) = \frac{-6(x-1)^2}{x^3(x-3)^2}$$

   Deduce that the curve with equation $y = f(x)$ has no turning points but has a horizontal point of inflection ; give the coordinates of this point.

   List the asymptotes of the curve and sketch it. (You may assume that there is no other point of inflection).
   Using the sketch find the real numbers **c** for which the set $\{x: f(x) = \mathbf{c}\}$ has three elements, find the number of elements in the set $\{x: f(x) = -x\}$

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4. Find a) $\int \frac{6x+5}{3x^2+5x+1}$ dx

b) $\int_2^8 \frac{dx}{(5x+2)^3}$ **3 , 4**

5. A car is travelling along a straight road. Its acceleration at time t seconds is

$$a(t) = 20 + 6t + 4t^2$$

measured in metres per second per second. The car started from rest at time t = 0 from a point O on the road. Find the speed of the car and its distance from O after t seconds.

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6. On [0 , 2] a function $y = f(x)$ is defined by $y = x^2 + 3$. Find the volume of the solid of revolution generated by revolving the area under f around the x-axis.

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7. Use Gaussian elimination to solve the system of equations.

$$\begin{aligned} 3x + 2y - z &= 9 \\ 2x - 3y - 2z &= 0 \\ x + 3y + 5z &= -3 \end{aligned}$$

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END OF QUESTION PAPER

ADVANCED HIGHER MATHEMATICS

$\frac{1}{46}$

UNIT 1 TEST solution.

1. (i) $\frac{d}{dx}e^{\cos x} = -\sin x . e^{\cos x}$ ✓ for Chain Rule ✓ for answer $\frac{}{2}$

(ii) $\frac{d}{dx}\frac{\sin x}{1+\cos^2 x} = \frac{(1+\cos^2 x).\cos x + \sin^2 x.2\cos x}{(1+\cos^2 x)^2}$ ✓ for Quotient Rule

$= \frac{\cos x(1+\cos^2 x + 2\sin^2 x)}{(1+\cos^2 x)^2}$ ⟵ ✓ for ~~expansion~~ substitut

$= \frac{\cos x(2+\sin^2 x)}{(1+\cos^2 x)^2}$ ✓ for factorising

✓ for solution. $\frac{}{4}$

2. $\left(2x^2 - \frac{3}{x}\right)^5 = \sum_{r=0}^{5}\binom{5}{r}(2x^2)^{5-r}\left(-\frac{3}{x}\right)^r$ ✓

$= \sum_{r=0}^{5}\binom{5}{r}(2)^{5-r}x^{10-2r}(-3)^r x^{-r}$

$= \sum_{r=0}^{5}\binom{5}{r}2^{5-r}(-3)^r x^{10-3r}$ ✓

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

✓ coeffs.

$= 1(2)^5x^{10} + 5(2)^4(-3)x^7$ ✓
$+ 10(2)^3(-3)^2x^4 + 10(2)^2(-3)^3x$
$+ 5(2)(-3)^4x^{-2} + 1(-3)^5x^{-5}$

$= 32x^{10} - 240x^7 + 720x^4$
$-1080x + \frac{810}{x^2} - \frac{243}{x^5}$ ✓

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3. $$f(x)=\frac{3x-1}{x^2(x-3)}$$

✓ for Quotient R

$$f'(x)=\frac{x^2(x-3).3-(3x-1)(3x^2-6x)}{x^4(x-3)^2}=\frac{-6x^3+12x^2-6x}{x^4(x-3)^2}$$

✓ for subst. & simplify

$$=\frac{-6x(x^2-2x+1)}{x^4(x-3)^2}=\frac{-6(x-1)^2}{x^3(x-3)^2}$$

✓ for factorise & cancel.

$$f'(x)=0 \Leftrightarrow (x-1)^2=0 \Leftrightarrow \underline{x=1}$$

$$\left.\underline{f(1)=-1}\right\}$$ ✓ BOTH needed. (1, -1)

x	$1-$		$1+$
$f'(x)$	$-$	0	$-$
$f(x)$	\	$-$	\

Hence (1, -1) is P.H.I. ✓ ALL needed

Asymptotes

x= 0 and x = 3 ✓ are vertical asymptotes.

$x \to 0- \Rightarrow f(x) \to +\infty$ $\qquad$ $x \to 3- \Rightarrow f(x) \to -\infty$

$x \to 0+ \Rightarrow f(x) \to +\infty$ $\qquad$ $x \to 3+ \Rightarrow f(x) \to +\infty$

$$f(x)=\frac{3x-1}{x^3-3x^2}=\frac{\frac{3}{x^2}-\frac{1}{x^3}}{1-\frac{3}{x}}\to 0 \text{ as } x\to\infty \Rightarrow x-$$ ✓ axis is a horizontal asymptote.

As $x \to +\infty \Rightarrow f(x) \to 0^+$ and $x \to -\infty \Rightarrow f(x) \to 0^+$

Also $f(x)=0 \Leftrightarrow 3x-1=0 \Leftrightarrow x=\frac{1}{3} \Leftrightarrow (\frac{1}{3},0)$ lies on line. ✓

see end of paper.

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The set $\{x : f(x) = c\}$ has three elements for c > 0. (Horizontal line on graph.) ✓

The set $\{x : f(x) = -x\}$ has four elements. (Sloping line on graph.) ✓

4. (a) $\int \frac{6x+5}{3x^2+5x+1}dx$ let $u = 3x^2 + 5x + 1$ ✓

$\frac{du}{dx} = 6x + 5$ $\int \frac{du}{u}$ ✓

using $\int \frac{f'(x)}{f(x)}dx$ ans is $\log(3x^2 + 5x + 1) + c$ ✓ 3

(b) $\int_2^8 \frac{dx}{(5x+2)^2}$ let $u = 5x + 2 \Leftrightarrow du = 5dx \Leftrightarrow dx = \frac{1}{5}du$

$= \frac{1}{5}\int_{12}^{42} \frac{du}{u^3}$ ✓

x	2	8
u	12	42

✓

$= \frac{1}{5}\left[-\frac{1}{2}u^{-2}\right]_{12}^{42}$ ✓

$= -\frac{1}{10}\left[\frac{1}{u^2}\right]_{12}^{42}$

$= \frac{1}{1568}$ ✓ 4.

5. $a(t) = 20 + 6t + 4t^2$

$v(t) = 20t + 3t^2 + \frac{4}{3}t^3 + c$ ✓ when v=0 and t=0 $\Rightarrow$ c=0 ✓

$v(t) = 20t + 3t^2 + \frac{4}{3}t^3$

$s(t) = 10t^2 + t^3 + \frac{t^4}{3} + c$ ✓ when s=0 and t=0 $\Rightarrow$ c=0 ✓

So $s(t) = 10t^2 + t^3 + \frac{t^4}{3}$ ✓ 5

6. $V = \pi \int_a^b y^2 \, dx$ $\qquad y^2 = (x^2 + 3)^2$

$$V = \pi \int_0^2 x^4 + 6x^2 + 9 \, dx \Rightarrow \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_0^2 = \frac{202}{5}\pi \text{ cubic units.}$$

or equivalent

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7.
$$\begin{bmatrix} 3 & 2 & -1 & 9 \\ 2 & -3 & -2 & 0 \\ 1 & 3 & 5 & -3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3 & 5 & -3 \\ 0 & -9 & -12 & 6 \\ 0 & 0 & -60 & 120 \end{bmatrix}$$

$$-60z = 120 \Rightarrow z = -2 \therefore -9y + 24 = 6 \Rightarrow y = 2 \therefore x + 6 - 10 = -3 \Rightarrow x = 1$$

End of paper