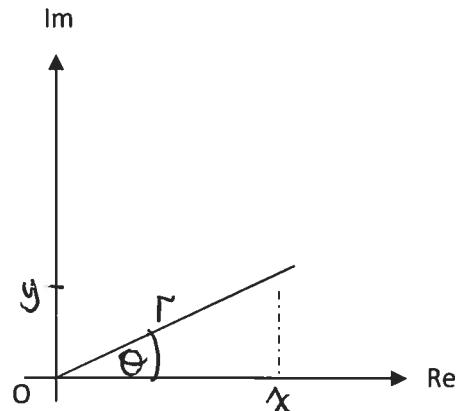


COMPLEX NUMBERS

The Argand Diagram

The complex number $x + iy$ is represented on the plane by the point $P(x, y)$.

The plane is referred to as the complex plane and diagrams of this sort are called Argand Diagrams.



The modulus of z , written $|z|$ is the distance between z and the origin.

$$|z| = \sqrt{x^2 + y^2} = r \quad (r > 0)$$

The argument (θ) of $z = x + iy$ is the angle OZ makes with the positive real axis

$$\tan \theta = \frac{y}{x}$$

Note

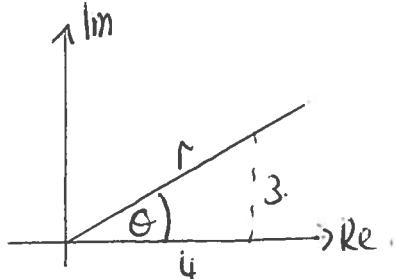
- the principle value of θ is $-\pi < \theta \leq \pi$
- $\arg z$ is used for short

~~$-\pi < \theta \leq \pi$~~
MISTAKE

Examples

1. Find the modulus and argument of $z = 4 + 3i$

* Always plot first so that you can get correct value for θ .



$$r = \sqrt{4^2 + 3^2}$$

$$r = 5$$

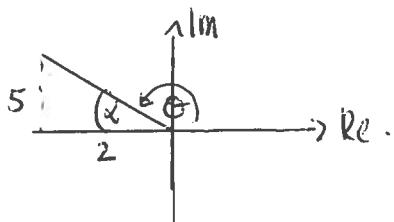
$$\tan \theta = \frac{3}{4}$$

$$\theta = 0.64 \text{ radians}$$

$$(or 36.9^\circ)$$

Note $\arg z$ can be in degrees or radians.

2. Find the modulus and argument of $z = -2 + 5i$



$$r = \sqrt{5^2 + 2^2}$$

$$r = \sqrt{29}$$

$$\tan \alpha = \frac{5}{2}$$

$$\alpha = 1.19$$

$$\text{so } \arg z = \theta$$

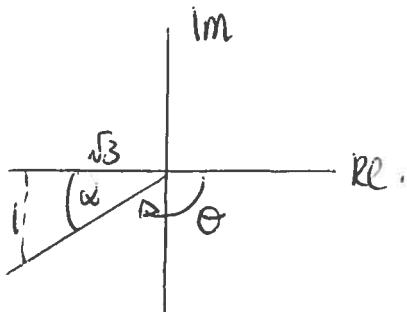
$$= \pi - 1.19$$

$$= 1.951 \text{ radians.}$$

Maths in Action Book 2 Page 94 Exercise 3 Questions 2, 3, 4

extra eg

- ③ Find the modulus and argument of $z = -\sqrt{3} - i$



$$r = \sqrt{3+1}$$

$$= 2$$

$$\tan \alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$\arg z = \theta = -(\pi - \frac{\pi}{6})$$

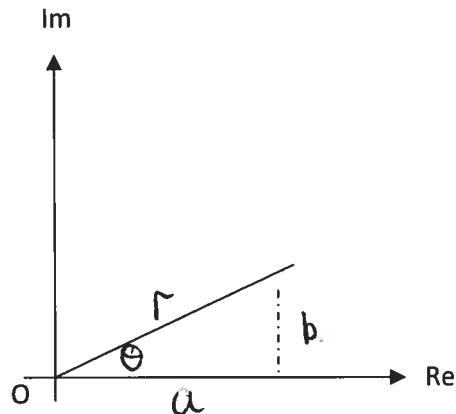
$$= -\frac{5\pi}{6}$$

Cartesian and Polar Form

$z = a + ib$ is in **cartesian form**

Note that

$$\begin{aligned}\sin \theta &= \frac{b}{r} & \cos \theta &= \frac{a}{r} \\ b &= r \sin \theta & a &= r \cos \theta\end{aligned}$$

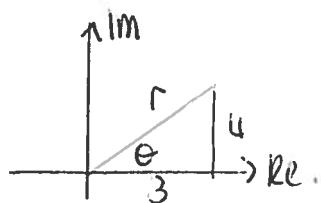


so $z = r(\cos \theta + i \sin \theta)$

$z = r(\cos \theta + i \sin \theta)$ is in **polar form**

Examples

1. Express $3 + 4i$ in polar form.



Find r and θ first.

$$r = \sqrt{4^2 + 3^2}$$

$$r = 5$$

$$\tan \theta = \frac{4}{3}$$

$$\theta = 0.927 \text{ radians}$$

$$z = 5(\cos 0.927 + i \sin 0.927)$$

2. Express $z = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$ in cartesian form.

$$z = 2\left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right)$$

$$z = \sqrt{3} + i$$

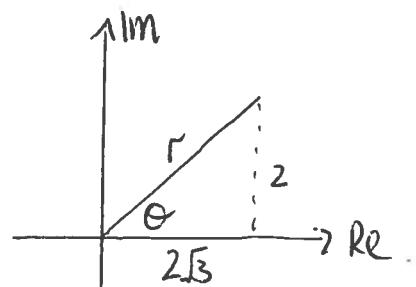
3. Express $\frac{-8}{i-\sqrt{3}}$ in polar form.

Change to Cartesian form first
(i.e. $(a+ib)$)

$$\begin{aligned}\frac{-8}{i-\sqrt{3}} &= \frac{-8(-i-\sqrt{3})}{(i-\sqrt{3})(-i-\sqrt{3})} \\ &= \frac{8i + 8\sqrt{3}}{-i^2 - \sqrt{3}i + i\sqrt{3} + 3} \\ &= \frac{8\sqrt{3} + 8i}{4} \\ &= 2\sqrt{3} + 2i\end{aligned}$$

$$\begin{aligned}r &= \sqrt{(2\sqrt{3})^2 + 2^2} \\ &= \sqrt{12+4} \\ &= 4\end{aligned}$$

$$\frac{-8}{i-\sqrt{3}} = 4 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$



$$\tan \theta = \frac{2}{2\sqrt{3}}$$

$$\tan \theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

Tips for success

- remember $-\pi < \arg z \leq \pi$
- plot z on a graph diagram to find r and θ .
- get z in form $a+ib$ first before writing in polar form.

Sets of Points (Loci) on the Complex Plane

A locus is a set of points which satisfies a given condition.

Example

- Given $z = x + iy$ draw the locus of the point on the complex plane such that

$$|z - 2| = 4$$

$$|z - 2| = 4$$

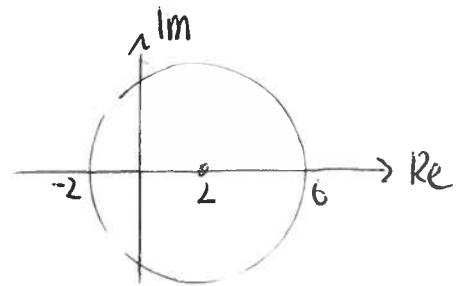
$$|(x+iy) - 2| = 4$$

$$|(x-2) + iy| = 4$$

$$\sqrt{(x-2)^2 + y^2} = 4$$

$$(x-2)^2 + y^2 = 16$$

circle centre $(2, 0)$ radius 4



- If $z = x+iy$ find the equation of the locus $\arg z = \frac{\pi}{3}$

$$\arg z = \frac{\pi}{3}$$

$$\tan \frac{\pi}{3} = \frac{y}{x}$$

$$\frac{y}{x} = \sqrt{3}$$

$$y = \sqrt{3}x$$

straight line through 0
gradient $\sqrt{3}$.

3. Find the locus for $|z - 1| = |z - i|$

$$\text{let } z = x + iy$$

$$|(x+iy) + 1|^2 = |x + iy - i|^2$$

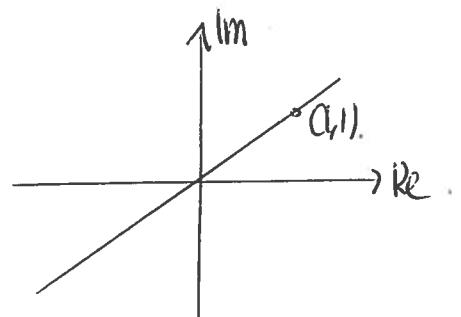
$$|(x-1) + iy|^2 = |x + (y-1)i|^2$$

$$(x-1)^2 + y^2 = x^2 + (y-1)^2$$

$$x^2 + 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$$

$$2x = 2y$$

$$x = y$$



Tips for success

- To find a locus let $z = x + iy$ in given condition and find equation connecting x and y .

Multiplication and Division in Polar Form

Results

multiplying

$$|z_1 z_2| = |z_1| |z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

dividing

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

When multiplying
multiply modulus
add arguments.

When dividing
divide modulus
subtract arguments.

Proof

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

Multiply

$$\begin{aligned} z_1 z_2 &= r_1 (\cos \theta_1 + i \sin \theta_1) r_2 (\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

Thus $|z_1 z_2| = r_1 r_2 = |z_1| |z_2|$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

Dividing

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} \\ &= \frac{r_1}{r_2} \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \\ &= \frac{r_1}{r_2} \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \\ &= \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \end{aligned}$$

Thus $\left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2}$ $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

Example

- Simplify $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \times 5\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
 $= 20 \left(\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)\right)$

\nearrow multiply modulus. \nwarrow add arguments

 $= 20 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ ← polar form
 $= 20i$ ← cartesian form

- Simplify $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \div 2\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)$
 $= 4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \div 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$
 $= 2 \left(\cos\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right) + i \sin\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)\right)$
 $= 2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$
 $= 2i$

For polar form must be + here.
 remember $\cos\left(-\frac{\pi}{6}\right) = \cos \frac{\pi}{6}$
 $\sin\left(-\frac{\pi}{6}\right) = -\sin \frac{\pi}{6}$.
 so $\cos \frac{\pi}{6} = i \sin \frac{\pi}{6}$
 $= \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$

Tips for success

- make sure numbers are in polar form before
 using $|z_1 z_2| = |z_1||z_2|$ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
 $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.
- remember argument must be between $-\pi$ and π
 so may need to adjust angle by adding or subtracting 2π .

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ then

$$z^n = r^n (\cos\theta + i\sin\theta)^n = r^n (\cos n\theta + i\sin n\theta)$$

We can prove this for $n \geq 1, n \in N$ - see proof notes

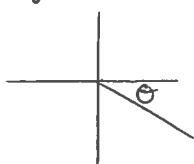
In fact it is true for all $n \in R$

Examples

1. Evaluate $(1-i)^7$ (a) in polar form (b) in cartesian form

$$\text{let } z = (1-i)$$

Change to polar form



$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{-1}{1}$$

$$\theta = -\frac{\pi}{4}$$

ra $\frac{\pi}{4}$

4th quad

$$\text{so } z = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \right)$$

$$(a) z^7 = (\sqrt{2})^7 \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \right)^7$$

$$= 8\sqrt{2} \left(\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right) \right) \quad \text{make angle in range } (-\pi, \pi]$$

$$(b) z^7 = 8\sqrt{2} \left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) \right)$$

$$2. \text{ Evaluate } z = \left(4 \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \right) \right)^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}} \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \right)$$

$$= 2 \left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right) \right)$$

$$= \sqrt{3} + i$$



Tips for success

- Use de Moivre's Theorem for powers of a complex number \rightarrow write number in polar form first.
 - remember final answer in polar form must have angle which is between $-\pi$ and π .

Maths in Action Book 2 Page 101 Exercise 6 Questions 1, 2

3. Using De Moivre's Theorem express $\sin 5\theta$ in terms of $\sin \theta$.

De Moivre's theorem gives

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta.$$

Using binomial theorem

$$(\cos\theta + i\sin\theta)^5$$

$$\cos 5\theta + i \sin 5\theta = \cos 5\theta + 5 \cos^4 \theta i \sin \theta + 10 \cos^3 \theta i^2 \sin^2 \theta \\ + 10 \cos^2 \theta i^3 \sin^3 \theta + 5 \cos \theta i^4 \sin^4 \theta + i^5 \sin^5 \theta.$$

			1
		1	1
L	2	1	
1	3	3	1
1	4	6	4
1	5	10	10
		5	1

Private imaginary part

$$\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta.$$

$$= 5(1-\sin^2\theta)^2 \sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta.$$

$$= 5\sin\theta (1 - 2\sin^4\theta + \sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^7\theta + 10\sin^9\theta + \sin^{10}\theta.$$

$$80 \quad \sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

Note To get $\cos 5\theta$ we equate real parts instead.

$$\begin{aligned}
 \cos 5\theta &= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta \\
 &= \cos^5\theta - 10\cos^3\theta (1-\cos^2\theta) + 5\cos\theta (1-\cos^2\theta)^2 \\
 &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta (1-2\cos^2\theta + \cos^4\theta) \\
 &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\
 &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .
 \end{aligned}$$

Tips for success

- to express $\sin n\theta$ and/or $\cos n\theta$ in terms of $\sin\theta$ or $\cos\theta$ consider
 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
 ↳ expand using Binomial theorem
- equate real and imaginary parts.
 Use $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$.

4. If $z = -8i$ find the cube roots of z .

Note z in polar form

$$|z| = \sqrt{8^2}$$

$$= 8$$

$$\arg z = -\frac{\pi}{2}$$

$$z = 8 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)$$

$$\text{We want } z^{\frac{1}{3}} = 8^{\frac{1}{3}} \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)^{\frac{1}{3}}$$

* This will give only one solution

\rightarrow there should be 3 cube roots (if using complex numbers)

\rightarrow use other values of argument

$$\text{i.e. } -\frac{\pi}{2} + 2n\pi$$

\nwarrow adding on multiples of 2π gives more solutions.

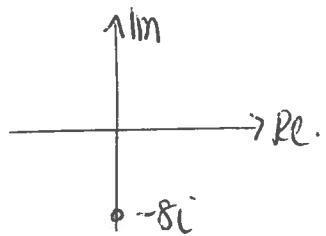
We consider

$$z = 8 \left(\cos \left(-\frac{\pi}{2} + 2n\pi \right) + i \sin \left(-\frac{\pi}{2} + 2n\pi \right) \right)$$

In this case:

$$z^{\frac{1}{3}} = 2 \left(\cos \left(-\frac{\pi}{2} + 2n\pi \right) + i \sin \left(-\frac{\pi}{2} + 2n\pi \right) \right)^{\frac{1}{3}}$$

$$\begin{aligned} \text{When } n=0 \quad z^{\frac{1}{3}} &= 2 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)^{\frac{1}{3}} \\ &= 2 \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right) \\ &= 2 \left(\frac{\sqrt{3}}{2} - i \left(\frac{1}{2} \right) \right) \\ &= \sqrt{3} - i \end{aligned}$$



When $n=1$

$$z^{\frac{1}{3}} = 2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)^{\frac{1}{3}}$$

$$\begin{aligned} z^{\frac{1}{3}} &= 2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\ &= 2i \end{aligned}$$

When $n=2$

$$\begin{aligned} z^{\frac{1}{3}} &= 2 \left(\cos \frac{7\pi}{2} + i \sin \frac{7\pi}{2} \right)^{\frac{1}{3}} \\ &= 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) \\ &= 2 \left(\left(-\frac{\sqrt{3}}{2} \right) + i \left(-\frac{1}{2} \right) \right) \\ &= -\sqrt{3} - i \end{aligned}$$

There are 3 distinct roots $\sqrt[3]{-1}$, $2i$ and $-\sqrt{3} - i$

Note Further values of n i.e. $n=3, 4, 5, \dots$ etc would just give the same roots again.

Note

Solutions to equations of the form $z^n = 1$ are called the n^{th} roots of unity

Example

Solve the equation $z^5 = 1$ to find the fifth roots of unity.

$$z^5 = 1$$

Write $z^5 = 1$ in polar form

$$|z^5| = 1 \quad \arg z^5 = 0$$

$$z^5 = 1 (\cos(0 + 2\pi n) + i \sin(0 + 2\pi n))$$

$$z = 1^{\frac{1}{5}} (\cos 2\pi n + i \sin(2\pi n))^{\frac{1}{5}}$$

$$\text{let } n=0 \quad z = 1 (\cos 0 + i \sin 0)^{\frac{1}{5}}$$

$$= 1$$

$$\text{let } n=1 \quad z = 1 (\cos 2\pi + i \sin 2\pi)^{\frac{1}{5}}$$

$$= \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$$

$$\text{let } n=2 \quad z = (\cos 4\pi + i \sin 4\pi)^{\frac{1}{5}}$$

$$= \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$$

$$\text{let } n=3 \quad z = (\cos 6\pi + i \sin 6\pi)^{\frac{1}{5}}$$

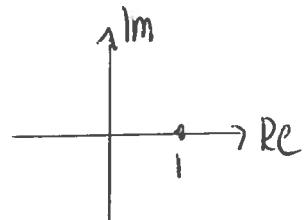
$$= \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$$

$$= \cos \left(-\frac{4\pi}{5}\right) + i \sin \left(-\frac{4\pi}{5}\right)$$

$$\text{let } n=4 \quad z = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$$

$$= \cos \left(-\frac{2\pi}{5}\right) + i \sin \left(-\frac{2\pi}{5}\right)$$

Solutions 1, $\cos \left(\pm \frac{2\pi}{5}\right) + i \sin \left(\pm \frac{2\pi}{5}\right)$, $\cos \left(\pm \frac{4\pi}{5}\right) + i \sin \left(\pm \frac{4\pi}{5}\right)$



remember argument
between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$

Tips for success

If asked to find the n th roots of unity solve

$$z^n = 1$$

→ write one in polar form with
+ $2n\pi$ for angle.

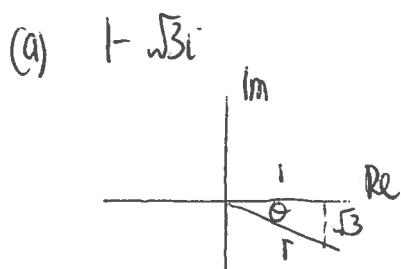
Then find n th root.

Maths in Action Book 2 Page 106 Exercise 7**Exam Question**

(a) Express $1 - \sqrt{3}i$ in polar form.

(b) Use de Moivre's theorem to show that $(1 - \sqrt{3}i)^n = 2^n \left(\cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right)$.

(c) Given that $z = 1 - \sqrt{3}i$ is a root of the equation $z^6 + 4(1 - i)z^3 + p + qi = 0$, where p and q are real numbers, find the values of p and q .



$$r = \sqrt{1^2 + (-\sqrt{3})^2}$$

$$r = 2$$

$$\tan \theta = -\frac{\sqrt{3}}{1}$$

$$\theta = -\frac{\pi}{3}$$

$$(1 - \sqrt{3}i) = 2 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right)$$

$$\begin{aligned}
 (b) \quad (1 - \sqrt{3}i)^n &= 2^n \left(\cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \right)^n \\
 &= 2^n \left(\cos\left(-\frac{n\pi}{3}\right) + i \sin\left(-\frac{n\pi}{3}\right) \right) \\
 &= 2^n \left(\cos\left(\frac{n\pi}{3}\right) - i \sin\left(\frac{n\pi}{3}\right) \right) \quad \text{as required.}
 \end{aligned}$$

$\cos(-x) = \cos x$
 $\sin(-x) = -\sin x$.

(c) $(1 - \sqrt{3}i)$ is a root so

$$(1 - \sqrt{3}i)^6 + 4(1-i)(1 - \sqrt{3}i)^3 + p + qi = 0 \quad (\star)$$

Using result in (b)

$$\begin{aligned}
 (1 - \sqrt{3}i)^6 &= 2^6 \left(\cos 2\pi - i \sin 2\pi \right) \\
 &= 64
 \end{aligned}$$

$$\begin{aligned}
 (1 - \sqrt{3}i)^3 &= 2^3 \left(\cos \pi - i \sin \pi \right) \\
 &= -8
 \end{aligned}$$

$$\text{In } (\star) \quad 64 + 4(1-i)(-8) + p + qi = 0$$

$$64 - 32(1-i) + p + qi = 0$$

$$32 + 32i + p + qi = 0$$

$$p + qi = -32 - 32i$$

$$p = -32 \text{ and } q = -32.$$