

Cfe AH Maths Practice Prelim Paper 2

All questions should be attempted

1. Given that $f(x) = \cos^{-1}(\sqrt{x})e^{\frac{1+x}{1-x}}$, $0 < x < 1$, obtain and simplify $f'(x)$. 4

2. Find the term independent of u in the expansion of $\left(\frac{2}{u^3} - 3u\right)^8$. 4

3. (a) Express the function $f(x) = \frac{6x^4 + x^3 - 5x - 4}{x^3 - x}$ in the form $Ax + B + \frac{C}{x} + \frac{D}{x+1} + \frac{E}{x-1}$, where A, B, C, D and E are integers. 4

 (b) Hence show that $\int_2^3 f(x) dx = 16 + \ln 6$ 4

4. (a) Express $z = \frac{4-2i}{3+i} - (1-2i)(3+i)$ in the form $x+iy$ where x and y are real numbers. 3

 (b) (i) Show that $-2i$ is a root of the equation $z(z^3 + 8z^2 + 36z + 32) + 128 = 0$. 2

 (ii) Write down a second root of the equation $z(z^3 + 8z^2 + 36z + 32) + 128 = 0$. 1

 (iii) Find the other two roots of the equation $z(z^3 + 8z^2 + 36z + 32) + 128 = 0$. 3

5. Prove by induction that $6^n - 1$ is divisible by 5 for all natural numbers n . 5

6. A curve is defined by the parametric equations

$$x = 10t, \quad y = 1 + 12t - t^3 \quad \text{for all } t.$$
 - (a) Find the coordinates of the stationary points of this curve. 4

 - (b) Obtain an expression for $\frac{d^2y}{dx^2}$ and use this to determine the nature of the stationary points found in (a). 3

7. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 8 - 5x.$$

Find the particular solution corresponding to the initial conditions $y = 5$ and $\frac{dy}{dx} = -6$ when $x = 0$.

10

8. The function g is defined by $g(x) = x - 2 + \frac{9}{(x+2)}$, $x \neq -2$, $x \in R$.

- (a) (i) Write down the equations of the asymptotes of g .
2
- (ii) Find the coordinates of the point where the graph of $y = g(x)$ crosses the y -axis.
1

- (b) Find the coordinates and nature of the stationary points of g .
5

- (c) Sketch the graph of $y = g(x)$, indicating the features found in (a) and (b).
3

9. (a) The sum of the first n terms of a series is given by $S_n = n^2 - 6n$.
 Find an expression for the n^{th} term, u_n .
3

- (b) Obtain the sum of the series $64 - 32 + 16 - \dots + \frac{1}{4}$.
3

10. Use the substitution $5x = 4\cos\theta$ to show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{16 - 25x^2} dx = \frac{4\pi + 6\sqrt{3}}{15}$$

(Note that $\cos 2A = 1 - 2\sin^2 A$.)

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CFe AH Maths Practice Prelim Paper 2 Solutions

$$\textcircled{1} \quad f(x) = \cos^{-1}(\sqrt{x}) e^{\frac{4+x}{4-x}}$$

Mark

first $f'(x) = u \frac{dv}{dx} + v \frac{du}{dx}$ ✓ product rule

$$u = \cos^{-1} x^{\frac{1}{2}}$$

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-(x^{\frac{1}{2}})^2}} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$v = e^{\frac{4+x}{4-x}}$$

$$\frac{du}{dx} = e^{\frac{4+x}{4-x}} \left(\frac{4-x-(4+x)(-1)}{(4-x)^2} \right)$$

$$= \frac{8}{(4-x)^2} e^{\frac{4+x}{4-x}}$$

$$\begin{aligned} \text{So } f'(x) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= \cos^{-1} x^{\frac{1}{2}} \cdot \frac{8}{(4-x)^2} e^{\frac{4+x}{4-x}} + e^{\frac{4+x}{4-x}} \cdot -\frac{1}{2\sqrt{x}\sqrt{1-x}} \\ &= \frac{8 \cos^{-1} \sqrt{x} e^{\frac{4+x}{4-x}}}{(4-x)^2} - \frac{e^{\frac{4+x}{4-x}}}{2\sqrt{x}\sqrt{1-x}} \\ &= e^{\frac{4+x}{4-x}} \left(\frac{8 \cos^{-1} \sqrt{x}}{(4-x)^2} - \frac{1}{2\sqrt{x}\sqrt{1-x}} \right) \end{aligned}$$

$$\textcircled{2}. \quad \left(\frac{2}{u^3} - 3u \right)^8$$

General term ${}^8C_r \left(\frac{2}{u^3} \right)^{8-r} (-3u)^r$

(2)

$$= {}^8C_r 2^{(8-r)} u^{(-24+3r)} (-3)^r u^r$$

$$= {}^8C_r 2^{8-r} (-3)^r u^{-24+4r}$$

Term independent of $u \Rightarrow -24 + 4r = 0$
 $r = 6.$

When $r = 6$ term ${}^8C_6 2^2 (-3)^6 u^0$
 $= 81648$

(3) $f(x) = \frac{6x^4 + x^3 - 5x - 4}{x^3 - x}$

Divide out

$$\begin{array}{r} 6x + 1 \\ \hline x^3 - x \Big| 6x^4 + x^3 - 5x - 4 \\ 6x^4 - 6x^2 \\ \hline x^3 + 6x^2 - 5x - 4 \\ x^3 - x \\ \hline 6x^2 - 4x - 4 \end{array}$$

So $f(x) = 6x + 1 + \frac{6x^2 - 4x - 4}{x^3 - x}$

Write $\frac{6x^2 - 4x - 4}{x^3 - x} = \frac{6x^2 - 4x - 4}{x(x-1)(x+1)}$ as partial fractions

$$= \frac{C}{x} + \frac{D}{x+1} + \frac{E}{x-1}$$

(3)

$$= \frac{C(x+1)(x-1) + Dx(x-1) + Ex(x+1)}{x(x-1)(x+1)}$$

$$\text{So } 6x^2 - 4x - 4 = C(x+1)(x-1) + Dx(x-1) + Ex(x+1)$$

$$\text{let } x=0 \quad -4 = -C$$

$$C = 4.$$

$$\text{let } x=1 \quad 6-4-4 = 2E$$

$$-2 = 2E$$

$$E = -1$$

$$\text{let } x=-1 \quad 6+4-4 = 2D$$

$$2D = 6$$

$$D = 3.$$

$$\text{So } f(x) = 6x+1 + \frac{4}{x} + \frac{3}{x+1} - \frac{1}{x-1}$$

$$(b) \int_2^3 f(x) dx$$

$$= \int_2^3 \left(6x+1 + \frac{4}{x} + \frac{3}{x+1} - \frac{1}{x-1} \right) dx$$

$$= \left[\frac{6x^2}{2} + x + 4\ln x + 3\ln|x+1| - \ln|x-1| \right]_2^3$$

$$= (3 \cdot 9 + 3 + 4\ln 3 + 3\ln 4 - \ln 2) - (3 \cdot 4 + 2 + 4\ln 2 + 3\ln 3 - \ln 1)$$

$$= 30 + \ln \frac{3^4 \cdot 4^3}{2^4 \cdot 1^3} - \ln 2^4 \cdot 3^3$$

$$= 16 + \ln \frac{3^4 \cdot 4^3}{2^4 \cdot 2^3}$$

(4)

$$= 16 + \ln 6 \quad \text{as required.}$$

(4) (a) $z = \frac{4-2i}{3+i} - (1-2i)(3+i)$

$$= \frac{(4-2i)}{(3+i)} \cdot \frac{(3-i)}{(3-i)} - (3-6i+i+2)$$

$$= \frac{12-4i-6i-2}{9+1} - (5-5i)$$

$$= \frac{10-10i}{10} - 5+5i$$

$$= 1-i - 5+5i$$

$$= -4+4i$$

(b) (i) $z(z^3 + 8z^2 + 36z + 32) + 128 = 0$

Let $z = -2i$

$$\text{LHS} = (-2i)((-2i)^3 + 8(-2i)^2 + 36(-2i) + 32) + 128$$

$$= (-2i)(-8i^3 + 32i^2 - 72i + 32) + 128$$

$$= (-2i)(8i - 32 - 72i + 32) + 128$$

$$= (-2i)(-64i) + 128$$

$$= -128 + 128$$

$$= 0$$

So $z = -2i$ is a root of the equation

(ii) second root $z = 2i$

(iii) $z = -2i$ and $z = 2i$ are roots

$\Rightarrow z+2i$ and $z-2i$ are factors.

(5)

$$\begin{aligned}
 & (z+2)(z-2) \\
 &= (z^2 + 2z - 2z + 4) \\
 &= (z^2 + 4) \checkmark
 \end{aligned}$$

We have $z(z^3 + 8z^2 + 36z + 32) + 128 = 0$
 $z^4 + 8z^3 + 36z^2 + 32z + 128 = 0$

Divide by factor $z^2 + 4$

$$\begin{array}{r}
 z^2 + 8z + 32 \\
 \hline
 z^4 + 8z^3 + 36z^2 + 32z + 128 \\
 z^4 + 4z^2 \\
 \hline
 8z^3 + 32z^2 + 32z + 128 \\
 8z^3 + 32z \\
 \hline
 32z^2 + 128 \\
 32z^2 + 128 \\
 \hline
 0
 \end{array}$$

So other roots given by $z^2 + 8z + 32 = 0$

$$\begin{aligned}
 z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-8 \pm \sqrt{64 - 4 \times 1 \times 32}}{2} \\
 &= \frac{-8 \pm \sqrt{-64}}{2} \\
 &= \frac{-8 \pm 8i}{2} \\
 &= \underline{-4 \pm 4i}
 \end{aligned}$$

⑥

⑤ Prove $6^n - 1$ is divisible by 5 for all $n \in \mathbb{N}$

$$\underbrace{n=1}_{6^1 - 1}$$

$$= 5$$

which is divisible by 5, so true for $n=1$

Assume the result is true for $n=k$

i.e. $6^k - 1$ is divisible by 5

so $6^k - 1 = 5p$, for some $p \in \mathbb{N}$. (*)

Consider $n = k+1$

$$6^{k+1} - 1$$

$$= 6(6^k) - 1$$

$$= 6(6^k - 1) + 6$$

$$= 6(6^k - 1) + 5$$

$$= 6 \cdot 5p + 5 \quad (\text{from } *)$$

$$= 5(6p + 1) \quad \text{which is divisible by 5.}$$

So since result is true for $n=1$ and if result is true for $n=k$ this implies that result is true for $n=k+1$, then result must be true for all $n \in \mathbb{N}$.

⑥ $x = 10t$ $y = 1 + 12t - t^3$

$$(i) \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \frac{dy}{dt} = 12 - 3t^2, \quad \frac{dx}{dt} = 10$$

(7)

$$\frac{dy}{dx} = \frac{12-3t^2}{10}$$

For stationary points $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{12-3t^2}{10} = 0$$

$$t^2 = 4$$

$$t = \pm 2$$

When $t = 2$ $x = 20$

$$y = 1 + 24 - 8 \\ = 17.$$

point $(20, 17)$

When $t = -2$ $x = -20$

$$y = 1 - 24 + 8 \\ = -15$$

point $(-20, -15)$

$$\begin{aligned} (b) \quad \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= -\frac{6}{10} t \cdot \frac{1}{10} \\ &= -\frac{3t}{50}. \end{aligned}$$

When $t = 2$ $\frac{d^2y}{dx^2} < 0 \Rightarrow (20, 17)$ is a max TP

When $t = -2$ $\frac{dy}{dx^2} > 0 \Rightarrow (-20, -15)$ is a min TP.

$$⑦ \quad \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 5y = 8 - 5x \quad (*)$$

Auxiliary equation

$$\begin{aligned} m^2 + 2m + 5 &= 0 \\ m &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

complementary function $y = Ae^{-x} \cos 2x + Be^{-x} \sin 2x$.

Particular integral $y = Cx + D$,

$$\frac{dy}{dx} = C$$

$$\frac{d^2y}{dx^2} = 0$$

$$\text{Substitute in } (*) \quad 0 + 2C + 5(Cx + D) = 8 - 5x$$

$$2C + 5Cx + 5D = 8 - 5x$$

$$2C + 5D = 8$$

$$5C = -5 \Rightarrow C = -1$$

$$\text{so } D = 2.$$

Particular integral $y = -x + 2$. ✓

General solution: $y = Ae^{-x} \cos 2x + Be^{-x} \sin 2x - x + 2$.

Particular solution.

$$x=0 \quad y=5 \Rightarrow 5 = A + 2 \\ A = 3.$$

$$x=0, y=5, \frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = Ae^{-x}(-2\sin 2x) + \cos 2x(-Ae^{-x}) \\ + Be^{-x}2\cos 2x + \sin 2x(-Be^{-x}) - 1$$

Substitute

$$-6 = -A + 2B - 1$$

$$-6 = -3 + 2B - 1$$

$$2B = -2$$

$$B = -1$$

$$\text{So } y = 3e^{-x} \cos 2x - e^{-x} \sin 2x - x + 2.$$

(8) (i) $g(x) = \frac{x-2}{x+2}$

Vertical asymptotes.

$$x+2=0$$

$$\underline{x = -2}.$$

Approach	x	\rightarrow	-2	\rightarrow
$g(x)$		(-)	∞	(+)

Horizontal asymptotes

$$\text{as } x \rightarrow \infty \quad g(x) \rightarrow x-2$$

so $\underline{y = x-2}$ is a horizontal asymptote

Approach

$$x \rightarrow +\infty$$

$$g(x) \rightarrow x-2 + \text{a little bit}$$

cusp
above

$$x \rightarrow -\infty$$

$$g(x) \rightarrow x-2 - \text{a little bit}$$

cusp
below

(10)

(ii) Cuts y-axis $\Rightarrow x=0$

$$\text{so } g(0) = 0 - 2 + \frac{9}{2}$$

$$= \frac{5}{2}$$

point $(0, \frac{5}{2})$

$$\begin{aligned} (b) \quad g'(x) &= 1 - 9(x+2)^{-2} \\ &= 1 - \frac{9}{(x+2)^2} \end{aligned}$$

For stationary points $g'(x) = 0$

$$1 - \frac{9}{(x+2)^2} = 0$$

$$(x+2)^2$$

$$\frac{9}{(x+2)^2} = 1$$

$$9 = (x+2)^2$$

$$x+2 = \pm 3$$

$$x = 1 \text{ or } -5$$

$$y = 2, -10$$

points $(1, 2)$ and $(-5, -10)$

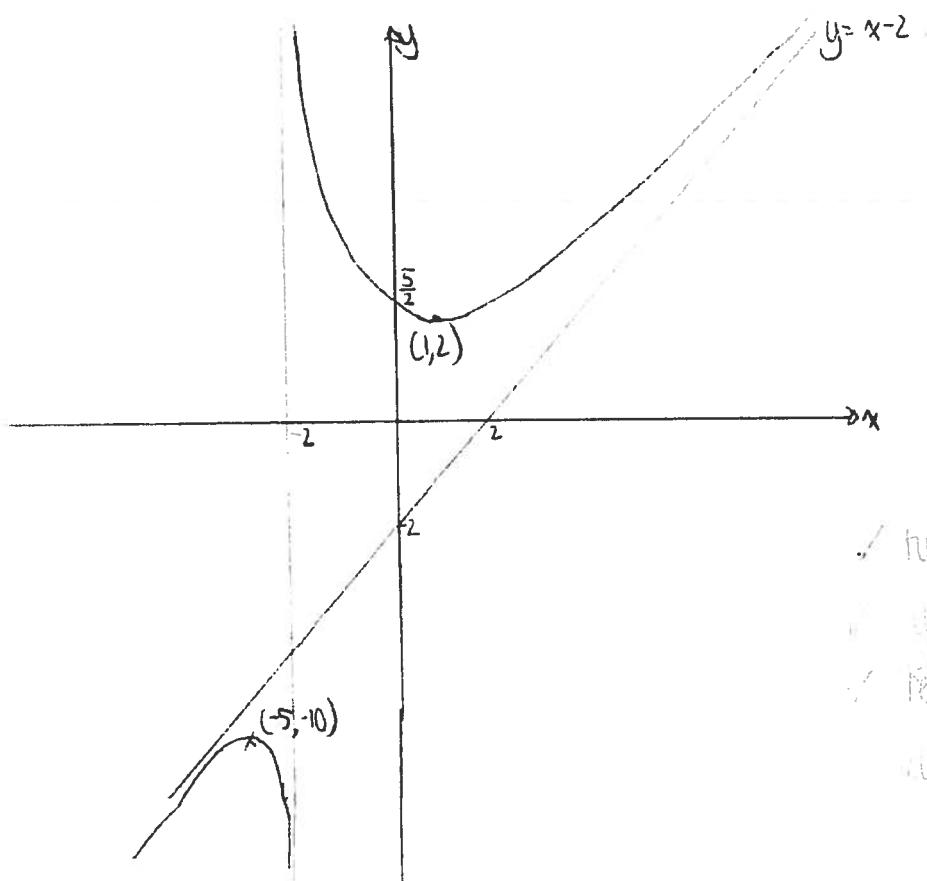
$$\text{Nature } g''(x) = 18(x+2)^{-3}$$

$$= \frac{18}{(x+2)^3}$$

When $x = 1 \quad g''(x) > 0 \Rightarrow (1, 2) \text{ is a min TP}$ $x = -5 \quad g''(x) < 0 \Rightarrow (-5, -10) \text{ is a max TP.}$

(11)

(c)



✓ function
✓ straight line
✓ different at
vertices

(3)

(9) (a)

$$S_n = n^2 - 6n$$

$$\begin{aligned} S_1 &= 1^2 - 6 \\ &= -5 \end{aligned}$$

$$\begin{aligned} S_2 &= 2^2 - 12 \\ &= -8 \end{aligned}$$

$$\begin{aligned} S_3 &= 3^2 - 18 \\ &= -9 \end{aligned}$$

$$\begin{aligned} S_4 &= 4^2 - 24 \\ &= -8 \end{aligned}$$

sequence -5, -3, -1, 1, ...

arithmetic $a = -5$ $d = 2$

$$\begin{aligned} u_n &= a + (n-1)d \\ &= -5 + (n-1) \cdot 2. \end{aligned}$$

$$u_n = 2n - 7.$$

(12)

$$(b) \quad 64 - 32 + 16 - \dots + \frac{1}{4}$$

Geometric $a = 64 \quad r = -\frac{1}{2}$

Find which term is $\frac{1}{4}$

$$U_n = ar^{n-1}$$

$$\frac{1}{4} = 64 \left(-\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{2^2} = 2^6 \left(-\frac{1}{2}\right)^{n-1}$$

$$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{2^8}$$

$$(-2)^{n-1} = 2^8$$

$$n-1 = 8$$

$$n = 9$$

$$\begin{aligned} \text{So } S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{64(1-\left(\frac{1}{2}\right)^9)}{1-(-\frac{1}{2})} \\ &= \underline{\underline{42.75}} \end{aligned}$$

$$\textcircled{D} \quad \int_0^{\frac{2}{5}} \sqrt{16 - 25x^2} dx$$

$$\text{let } 5x = 4 \cos \theta$$

$$x = \frac{4}{5} \cos \theta \quad \text{so} \quad \frac{dx}{d\theta} = -\frac{4}{5} \sin \theta$$

$$dx = -\frac{4}{5} \sin \theta d\theta.$$

$$\text{limits} \quad x=0 \quad \Rightarrow \quad \frac{4}{5} \cos \theta = 0$$

$$\theta = \frac{\pi}{2}$$

$$x = \frac{2}{5} \quad \Rightarrow \quad \frac{2}{5} = \frac{4}{5} \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Substitute

$$\begin{aligned}
 & \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{16 - 25 \left(\frac{4}{5} \cos \theta \right)^2} \cdot \left(-\frac{4}{5} \sin \theta d\theta \right) \\
 &= -\frac{4}{5} \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{16(1 - \cos^2 \theta)} \cdot \sin \theta d\theta. \\
 &= -\frac{16}{5} \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{1 - \cos^2 \theta} \cdot \sin \theta d\theta \\
 &= -\frac{16}{5} \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sin \theta \cdot \sin \theta d\theta \\
 &= -\frac{16}{5} \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sin^2 \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{16}{5} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} (1 - \cos 2\theta) d\theta. \\
&= -\frac{16}{5} \left[\frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}. \\
&= -\frac{16}{5} \left(\frac{\pi}{6} - \frac{1}{4} \sin \frac{2\pi}{3} \right) + \frac{16}{5} \left(\frac{\pi}{4} - \frac{1}{4} \sin \pi \right) \\
&= -\frac{16}{5} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} \right) + \frac{16}{5} \left(\frac{\pi}{4} \right) \\
&= -\frac{8\pi}{15} + \frac{2\sqrt{3}}{5} + \frac{4\pi}{5} \\
&= -\frac{8\pi}{15} + \frac{6\sqrt{3}}{15} + \frac{12\pi}{15} \\
&= \frac{4\pi + 6\sqrt{3}}{15} \quad \text{as required.} \quad (7)
\end{aligned}$$

Total 71