

Cfe AH Maths Practice Prelim Paper 1

All questions should be attempted

1. Given $x = \sin^{-1} t$, $y = \ln t$ where $0 < t < 1$, use parametric differentiation to obtain $\frac{dy}{dx}$ in terms of t . Simplify your answer. 3

2. Find the coefficient of x^4 in the expansion of $\left(\frac{x^2}{2} - \frac{4}{x^3}\right)^8$. 5

3. Express $\frac{x^2 - 2x + 6}{x^2(x^2 + 2)}$ in partial fractions. 5

- (4) Find the Maclaurin expansion of $\ln(1+x)$ as far as the term in x^4 . 3
 Given that the Maclaurin expansion of $\ln(\cos x)$ as far as the term in x^4 is $-\frac{x^2}{2} - \frac{x^4}{12}$, find the Maclaurin expansion as far as the term in x^4 of $\ln(\cos x + x \cos x)$. 3

5. Express $z = \frac{5i}{1+2i}$ in the form $a + ib$ where a and b are real numbers. 2
 Verify that z is a solution of the equation $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$ and find the other three roots. 4

6. Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all natural numbers n . 5

7. A curve is defined by the equation $x^3y^2 - 2xy + 8 = 0$, $x < 0$ and $y < 0$.
 Use implicit differentiation to find $\frac{dy}{dx}$. 3
 Hence find the equation of the tangent to the curve at the point where $x = -1$. 3

8. (a) Evaluate $\sum_{r=1}^3 16 \times \left(\frac{3}{4}\right)^{r-1}$ 2

- (b) Explain why the sum to infinity of the geometric series $16 + 12 + 9 + \dots$ exists and find this sum. 3

9. $I = \int_0^2 e^{\sqrt{4x+1}} dx$

- (a) Use the substitution $u = \sqrt{4x+1}$ to express I in the form $\int_a^b \frac{1}{k} ue^u du$, where a, b and k are integers. 3

- (b) Use integration by parts to evaluate the integral found in (a) 5

10. The function g is given by $g(x) = e^{-x} \sin 2x$.

- (a) Determine whether g is odd, even or neither. 2

- (b) Find the coordinates of the stationary point of g in the interval $0 < x < \frac{\pi}{2}$ 5

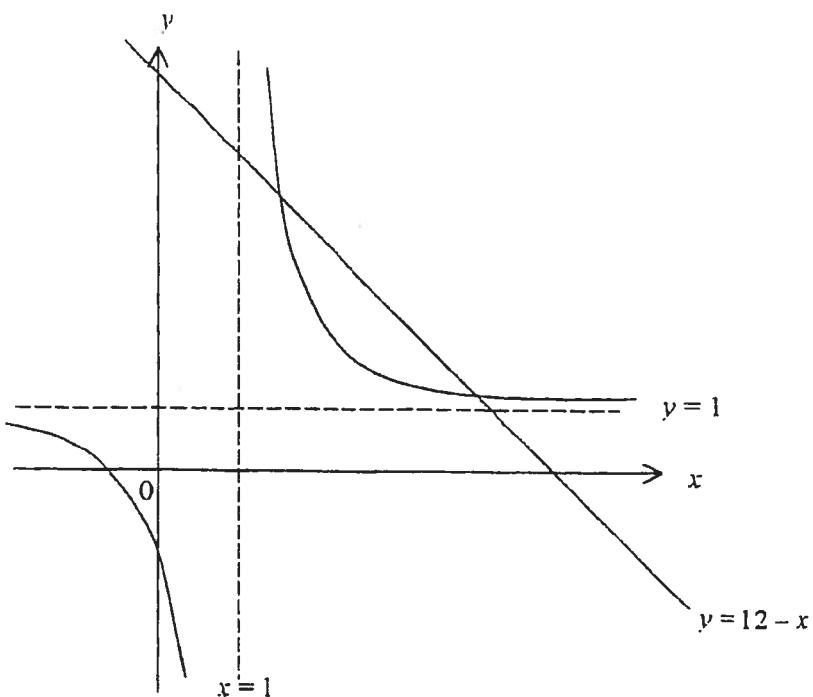
- (c) Obtain a formula for $y = g''(x)$. 1

- (d) Use your answer to (c) to determine the nature of the stationary point found in (b). 2

11. (a) Express $\frac{x+8}{x-1}$ in the form $A + \frac{B}{x-1}$. 2

- (b) The diagram below shows the curve with equation $y = \frac{x+8}{x-1}$ and the line with equation $y = 12 - x$.

Show that the shaded area can be written as $40 - \ln 9^2$. 6



(l)

Find the general solution of the differential equation

$$4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = 3x + 4.$$

Find the particular solution corresponding to the initial conditions $\frac{dy}{dx} = -3$

and $\frac{d^2y}{dx^2} = 4$ when $x = 0$.

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(Fe) Advanced Higher Prelim Practice Paper 1

$$\begin{aligned} \textcircled{1} \quad x &= \sin^{-1} t & y &= \ln t \\ \frac{dx}{dt} &= \frac{1}{\sqrt{1-t^2}} & \frac{dy}{dt} &= \frac{1}{t} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{1}{t} \cdot \frac{\sqrt{1-t^2}}{\sqrt{1-t^2}} \\ &= \frac{1}{t}. \end{aligned}$$

$$\textcircled{2} \quad \left(\frac{x^2}{2} - \frac{4}{x^3} \right)^8$$

$$\begin{aligned} \text{General Term} & \quad {}^8C_r \left(\frac{x^2}{2} \right)^{8-r} \left(-\frac{4}{x^3} \right)^r \\ &= {}^8C_r \left(\frac{1}{2} \right)^{8-r} x^{16-2r} (-4)^r x^{-3r} \\ &= {}^8C_r \left(\frac{1}{2} \right)^{8-r} (-4)^r x^{16-5r} \end{aligned}$$

$$\begin{aligned} \text{Coefficient of } x^{-4} & \quad 16-5r = -4 \\ & \quad -5r = -20 \\ & \quad r = 4. \end{aligned}$$

$$\begin{aligned} \text{coefficient} & \quad {}^8C_4 \left(\frac{1}{2} \right)^4 (-4)^4 \\ &= 1120. \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad \frac{x^2 - 2x + 6}{x^2(x^2+2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{(Cx+D)}{x^2+2} \\ &= \frac{Ax(x^2+2) + B(x^4+2) + (Cx+D)x^2}{x^2(x^2+2)} \end{aligned}$$

$$\text{Identity} \quad x^2 - 2x + 6 = Ax(x^2+2) + B(x^4+2) + ((Cx+D)x^2)$$

③ cont. Let $x = 0$ $6 = 2B$
 $B = 3.$

Let us Equate coefficients $x^3 \quad 0 = A + C$

$$x^2 \quad 1 = B + D$$

$$\Rightarrow D = -2.$$

$$x \quad -2 = 2A$$

$$A = -1 \quad \Rightarrow C = 1$$

constant $6 = 2B$

$$B = 3$$

So $\frac{x^2 - 2x + 6}{x^2(x^2+2)} = -\frac{1}{x} + \frac{3}{x^2} + \frac{x-2}{x^2+2}$

PTO

④

$$f(x) = \ln(1+x)$$

$$f(0) = 0$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f''(0) = -1$$

$$= -\frac{1}{(1+x)^2}$$

$$f'''(x) = 2(1+x)^{-3}$$

$$f'''(0) = 2$$

$$f^{(iv)}(x) = -6(1+x)^{-4}$$

$$f^{(iv)}(0) = -6$$

$$\begin{aligned} f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) \\ &= 0 + x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 \\ &= x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \dots \end{aligned}$$

$$\begin{aligned} &\ln(\cos x + x \cos x) \\ &= \ln(\cos x \cdot (1+x)) \\ &= \ln \cos x + \ln(1+x) \\ &= -\frac{x^2}{2} - \frac{x^4}{12} + x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 \\ &= x - x^2 + \frac{1}{3} x^3 - \frac{1}{3} x^4. \end{aligned}$$

$$\begin{aligned}
 ⑤ \quad z &= \frac{5i}{1+2i} \\
 &= \frac{5i(1-2i)}{(1+2i)(1-2i)} \\
 &= \frac{5i - 10i^2}{1 - 4i^2} \\
 &= \frac{10 + 5i}{5} \\
 &= 2 + i
 \end{aligned}$$

Let $z = 2+i$

$$\begin{aligned}
 z^4 - 4z^3 + 6z^2 - 4z + 5 &= (2+i)^4 - 4(2+i)^3 + 6(2+i)^2 - 4(2+i) + 5 \\
 &= 16 + 32i + 8i^2 \\
 &= 2^4 + 4 \cdot 2^3 i + 6 \cdot 2^2 i^2 + 4 \cdot 2i^3 + i^4 - 4(2^3 + 3 \cdot 2^2 i + 3 \cdot 2i^2) \\
 &\quad + 6(4 + 4i + i^2) - 8 - 4i + 5 \\
 &= 16 + 32i - 24 - 8i + 1 - 32 - 48i + 24 + 4i \\
 &= 0 \quad \text{so } z = 2+i \text{ is a root as required.}
 \end{aligned}$$

Hence $2-i$ is also a root.

$$\begin{aligned}
 &(z - (2-i))(z - (2+i)) \\
 &= z^2 - (2+i)z + z(2-i) + (4-i^2) \\
 &= z^2 - 4z + 5
 \end{aligned}$$

$$\begin{array}{r}
 z^2 + 1 \\
 \hline
 z^2 - 4z + 5 \quad | \frac{z^4 - 4z^3 + 6z^2 - 4z + 5}{z^4 - 4z^3 + 5z^2} \\
 \hline
 z^2 - 4z + 5
 \end{array}$$

Other roots given by $\frac{z^2 + 1}{z^2 - 1} = 0$

$$\begin{aligned}
 z^2 &= -1 \\
 z &= \pm i
 \end{aligned}$$

Roots are $z = 2+i, 2-i, i, -i$

⑥ When $n=1$

$$\begin{aligned}
 8^n - 7n + 6 &= 8^1 - 7 + 6 \\
 &= 7 \quad \text{which is divisible by 7} \\
 \text{Hence result is true for } n=1
 \end{aligned}$$

Assume result is true for $n=k$.

$$\text{i.e. } 8^k - 7k + 6 = 7m \quad , \quad m \in \mathbb{N}$$

Consider $n=k+1$

$$8^{k+1} - 7(k+1) + 6$$

From * $8^k = 7k - 6 + 7m$

$$\begin{aligned}
 \text{So} \quad & 8^{k+1} - 7(k+1) + 6 \\
 &= 8(7k - 6 + 7m) - 7(k+1) + 6 \\
 &= 56k - 48 + 56m - 7k - 7 + 6 \\
 &= 49k + 56m - 44 \\
 &= 7(7k + 8m - 7)
 \end{aligned}$$

which is divisible by 7.

So result holds for $n=k+1$

Since result is true for $n=1$ and if result is true for $n=k$ then it is true for $n=k+1$, by induction the result must be true for all $n \in \mathbb{N}$.

⑦ $x^3y^2 - 2xy + 8 = 0$

Differentiate.

$$x^3 \cdot 2y \frac{dy}{dx} + y^2 \cdot 3x^2 - \left(2x \frac{dy}{dx} + y \cdot 2 \right) = 0$$

$$2x^3y \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - 3x^2y^2$$

$$\frac{dy}{dx} = \frac{2y - 3x^2y^2}{2x^3y - 2x}$$

When $x = -1$ put in $x^3y^2 - 2xy + 8 = 0$
 $-y^2 + 2y + 8 = 0$
 $y^2 - 2y - 8 = 0$
 $(y-4)(y+2) = 0$
 $y = 4 \text{ or } y = -2.$
Since $y < 0 \Rightarrow \underline{\underline{y = -2}}$

Substitute in $\frac{dy}{dx} = \frac{-4 - 3x \times 4}{4 + 2}$
 $= \frac{-16}{6}$
 $= -\frac{8}{3}$

Equation $y - b = m(x - a)$ $m = -\frac{8}{3}$ $(a, b) = (-1, -2)$
 $y + 2 = -\frac{8}{3}(x + 1)$
 $3y + 6 = -8x - 8$
 $\underline{3y + 8x + 14 = 0}$

⑧ (a) $\sum_{r=1}^3 16x \left(\frac{3}{4}\right)^{r-1}$
 $= 16 \cdot \left(\frac{3}{4}\right)^0 + 16 \cdot \left(\frac{3}{4}\right)^1 + 16 \cdot \left(\frac{3}{4}\right)^2$
 $= 16 + 12 + 9$
 $= 37$

(b) common ratio $r = \frac{12}{16} = \frac{3}{4}$

Sum to infinity exists since $-1 < \frac{3}{4} < 1$

$$\begin{aligned}
 S_{\text{av}} &= \frac{a}{1-\Gamma} \\
 &= \frac{16}{1-\frac{3}{4}} \\
 &= 16 \times 4 \\
 &= \underline{\underline{64}}
 \end{aligned}$$

$$(q) \quad I = \int_0^2 e^{\sqrt{4x+1}} dx$$

$$\text{Let } u = \sqrt{4x+1} = (4x+1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4.$$

$$\frac{du}{dx} = \frac{2}{(4x+1)^{\frac{1}{2}}}$$

$$\frac{1}{2}(4x+1)^{\frac{1}{2}} du = dx. \quad \text{i.e.} \quad \frac{1}{2}u du = dx.$$

$$\text{limits when } x=0 \quad u = \sqrt{1}$$

$$\text{when } x=2 \quad u = \sqrt{9} = 3.$$

$$\begin{aligned}
 I &= \int_0^2 e^{\sqrt{4x+1}} dx \\
 &= \int_1^3 e^u \cdot \frac{1}{2} u du \\
 &= \int_1^3 \frac{1}{2} ue^u du \quad \text{as required.}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \int_1^3 \frac{1}{2} ue^u du &= \left[\frac{1}{2}u \cdot e^u \right]_1^3 - \int_1^3 e^u \cdot \frac{1}{2} du \\
 &= \left[\frac{3}{2}e^3 - \frac{1}{2}e \right]_1^3 - \frac{1}{2} \int e^u du \Big|_1^3
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{3}{2}e^3 - \frac{1}{2}e \right) - \frac{1}{2}(e^3 - e) \\
 &= \underline{\underline{e^3}}
 \end{aligned}$$

(10). (a) $g(x) = e^{2x} \sin 2x$

$$\begin{aligned}
 g(-x) &= e^{-2x} \sin(-2x) \\
 &= -e^{-2x} \sin 2x
 \end{aligned}$$

Since this is not $g(x)$ or $-g(x)$,

$g(x)$ is neither odd or even.

(b) $g(x) = e^{2x} \sin 2x$

$$\begin{aligned}
 g'(x) &= e^{2x} \cdot 2\cos 2x + \sin 2x \cdot 2e^{2x} \\
 &= 2e^{2x} (\cos 2x + \sin 2x)
 \end{aligned}$$

For stationary points $g'(x) = 0$

$$\Rightarrow 2e^{2x} (\cos 2x + \sin 2x) = 0$$

$$\cos 2x + \sin 2x = 0$$

$$\sin 2x = -\cos 2x$$

$$\tan 2x = -1$$

$$2x = \frac{3\pi}{4} \quad (\text{since } 0 < x < \frac{\pi}{2})$$

$$x = \frac{3\pi}{8}$$

By co-ordinate. Substitute $x = \frac{3\pi}{8}$ into.

$$g(x) = e^{2x} \sin 2x$$

$$\begin{aligned}
 g\left(\frac{3\pi}{8}\right) &= e^{\frac{3\pi}{4}} \sin \frac{3\pi}{4} \\
 &= \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}
 \end{aligned}$$

$$\text{Point } \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}} \right)$$

$$(c) \quad g'(x) = 2e^{2x} (\cos 2x + \sin 2x)$$

$$\begin{aligned} g''(x) &= 2e^{2x} (-2\sin 2x + 2\cos 2x) + (\cos 2x + \sin 2x) 4e^{2x} \\ &= -4e^{2x} \sin 2x + 4e^{2x} \cos 2x + 4e^{2x} \cos 2x + 4e^{2x} \sin 2x \\ &= 8e^{2x} \cos 2x. \end{aligned}$$

$$(d) \quad \text{When } x = \frac{3\pi}{8} \quad g''(x) = 8e^{\frac{3\pi}{4}} \cos \frac{3\pi}{4} \\ = -\frac{8}{\sqrt{2}} e^{\frac{3\pi}{4}} \\ < 0$$

so $x = \frac{3\pi}{8}$ gives a maximum T.P.

$$(II) \quad (a) \quad \frac{x+8}{x-1}$$

$$x-1 \overline{) \frac{x+8}{x-1}} \quad \begin{array}{r} | \\ x+8 \\ \hline x-1 \\ \hline 9 \end{array}$$

$$\frac{x+8}{x-1} = 1 + \frac{9}{x-1}$$

$$(b) \quad \text{Points of intersection} \quad \frac{x+8}{x-1} = 12-x$$

$$x+8 = (12-x)(x-1)$$

$$x+8 = 12x - 12 - x^2 + x$$

$$\begin{aligned} x^2 - 12x + 20 &= 0 \\ (x-10)(x-2) &= 0 \end{aligned}$$

$$x=10 \text{ or } x=2$$

$$\text{area} = \int_2^{10} \left((12-x) - \frac{x+8}{x-1} \right) dx$$

$$\int_2^{10} \left(12-x - 1 - \frac{9}{x-1} \right) dx$$

$$\begin{aligned}
 &= \left\{ \left(11x - \frac{x^2}{2} - 9 \ln|x-1| \right) \right\}_2^0 \\
 &= (110 - 50 - 9 \ln 9) - (22 - 2 - 9 \ln 1) \\
 &= \underline{\underline{40 - 9 \ln 9}} \\
 &= \underline{\underline{40 - \ln 9^9}} \quad \text{as required.}
 \end{aligned}$$

(12)

$$4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = 3x + 4$$

Auxiliary equation

$$4m^2 + 4m + 1 = 0$$

$$(2m+1)(2m+1) = 0$$

$$m = -\frac{1}{2}$$

$$\text{Complementary Function} \quad y = Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x}$$

Particular Integral

$$y = Cx + D$$

$$\frac{dy}{dx} = C$$

$$\frac{d^2y}{dx^2} = 0$$

$$\text{Substitute.} \quad 0 + 4C + (Cx + D) = 3x + 4$$

$$Cx + 4C + D = 3x + 4$$

$$C = 3$$

$$4C + D = 4$$

$$12 + D = 4$$

$$D = -8.$$

(12) cont.

P.I $y = 3x - 8$

General solution $y = CF + PI$

$$y = Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x} + 3x - 8$$

When $x=0 \quad \frac{dy}{dx} = -3.$

$$\frac{dy}{dx} = -\frac{1}{2}Ae^{-\frac{1}{2}x} + Bx \cdot -\frac{1}{2}e^{-\frac{1}{2}x} + e^{-\frac{1}{2}x} \cdot B + 3.$$

Substitute. $-3 = -\frac{1}{2}A + B + 3.$

$$B = A - 2B - 6$$

$$A - 2B = 12 \dots \textcircled{1}$$

When $x=0 \quad \frac{d^2y}{dx^2} = 4$

$$\frac{d^2y}{dx^2} = +\frac{1}{4}Ae^{-\frac{1}{2}x} - \frac{1}{2}Bx \cdot -\frac{1}{2}e^{-\frac{1}{2}x} - \frac{1}{2}Be^{-\frac{1}{2}x} - \frac{1}{2}Be^{-\frac{1}{2}x}$$

$$4 = \frac{1}{4}A - \frac{1}{2}B - \frac{1}{2}B$$

$$16 = A - 4B.$$

$$A - 4B = 16 \dots \textcircled{2}$$

$\textcircled{1} - \textcircled{2} \quad 2B = -4$

$$B = -2.$$

In $\textcircled{1} \quad A + 4 = 12$

$$A = 8.$$

Particular solution $y = 8e^{-\frac{1}{2}x} - 2xe^{-\frac{1}{2}x} + 3x - 8$