

## Cfe AH Maths Mini Prelim

### Practice Paper 2

**All questions should be attempted**

1.  $A = \begin{pmatrix} p & 1 \\ -2 & q \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$  and  $AB = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$

- (a) Find the values of  $p$  and  $q$ . 2
- (b) Hence find the matrix  $BA$ . 1
- (c) A matrix  $C$  is such that  $CAB = BA$ . Find the matrix  $C$ . 2

2.  $\dots$  6

Write the complex number  $z = -\sqrt{3} + i$  in polar form.

Hence:

- (a) express  $z^4$  in the form  $x + yi$ , where  $x$  and  $y$  are real numbers
- (b) show that  $z^6 + 64 = 0$ .

3. The lines  $L_1$  and  $L_2$  are given by

$$L_1 : \frac{x-1}{1} = \frac{y}{-3} = \frac{z+3}{4} \quad \text{and} \quad L_2 : \frac{x-4}{1} = \frac{y+5}{-1} = \frac{z-5}{2}$$

- (i) Show that  $L_1$  and  $L_2$  intersect and find the point of intersection. 4
- (ii) Find the size of the acute angle between  $L_1$  and  $L_2$ . 3
- (iii) Find the equation of the plane defined by  $L_1$  and  $L_2$ . 3

4. Prove by induction that  $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$  for all positive integers  $n$ . 5

5. Express the number  $432_5$  in base 7. 3

6. Obtain the first four non-zero terms in the Maclaurin expansion of  $\ln\left(1 - \frac{x}{2}\right)$ . 4

Hence show that  $\ln(2-x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4$ . 1

7. The function  $f$  is defined by  $f(x) = ax^3 + bx^2 + cx + 1$  where  $a, b$  and  $c$  are constants.

It is known that the graph of  $f$  passes through the point  $(1, 0)$  and has a stationary point at  $(-2, 9)$ .

- (a) Deduce that  $a, b$  and  $c$  must satisfy the system of equations

$$\begin{aligned}a + b + c &= -1 \\4a - 2b + c &= -4 \\12a - 4b + c &= 0.\end{aligned}$$

4

- (b) Use Gaussian elimination to find the values of  $a, b$  and  $c$ .

5

8. Two matrices  $A$  and  $B$  are given as

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & \alpha \\ 1 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 1 & 2 \\ -1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Find the value of  $\alpha$  for which the matrix  $3A - 2B$  is singular.

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[ END OF QUESTION PAPER ]

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## Solutions

$$\textcircled{1} \quad (a) \quad AB = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} p & 1 \\ -2 & q \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2p & p^{-1} \\ -4 & -2-q \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$$

$$2p = 2 \quad \checkmark \quad -2-q = 2 \quad \checkmark \\ p = 1 \quad \quad \quad q = -4.$$

$$(b) \quad BA = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix} \\ = \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} \quad \checkmark$$

$$(c) \quad CAB = BA$$

$$\text{so, } C = BA(AB)^{-1} \quad \checkmark$$

$$\det AB = \frac{1}{4} \quad (AB)^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$$

$$\text{so } C = \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \\ = \frac{1}{4} \begin{pmatrix} -8 & -4 \\ 20 & 8 \end{pmatrix}$$

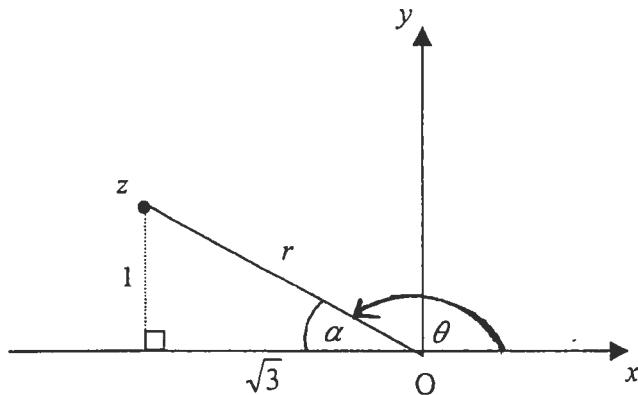
$$= \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$$

(2)

(2)

### Solution

$$z = -\sqrt{3} + i \rightarrow \text{plot } (-\sqrt{3}, 1)$$



$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\begin{aligned} \tan \alpha &= \frac{1}{\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ \\ &\Rightarrow \theta = 180^\circ - 30^\circ = 150^\circ \end{aligned}$$

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 2(\cos 150^\circ + i \sin 150^\circ) \end{aligned}$$

$$\begin{aligned} (a) \quad \text{By de Moivre's theorem:} \quad z^4 &= 2^4 \{ \cos(4 \times 150^\circ) + i \sin(4 \times 150^\circ) \} \\ &= 16(\cos 600^\circ + i \sin 600^\circ) \\ &= 16 \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \\ &= -8 - 8\sqrt{3}i \end{aligned}$$

$$\begin{aligned} (b) \quad \text{By de Moivre's theorem:} \quad z^6 &= 2^6 \{ \cos(6 \times 150^\circ) + i \sin(6 \times 150^\circ) \} \\ &= 64(\cos 900^\circ + i \sin 900^\circ) \\ &= 64(-1 + 0i) \\ &= -64 \end{aligned}$$

$$\text{Hence } z^4 + 64 = -64 + 64 = 0.$$

$$\textcircled{3} \quad (1) \quad l_1 \quad \frac{x-1}{1} = \frac{y}{-3} = \frac{z+3}{4}$$

$$l_2 \quad \frac{x-4}{1} = \frac{y+5}{-1} = \frac{z-5}{2}$$

$$l_1 \quad x = t+1 \\ y = -3t \\ z = 4t-3$$

$$l_2 \quad x = s+4 \\ y = -s-5 \\ z = 2s+5$$

$$\text{Intersection:} \quad t+1 = s+4 \quad \textcircled{1}$$

$$-3t = -s-5 \quad \textcircled{2}$$

$$4t-3 = 2s+5 \quad \textcircled{3}$$

$$\textcircled{1} + \textcircled{2} \quad -2t+1 = -1$$

$$-2t = -2$$

$$t = 1$$

$$\text{In } \textcircled{1} \quad 2 = s+4 \\ s = -2.$$

$$\text{Check in } \textcircled{3} \quad 4t-3 = 2s+5 \\ \Rightarrow 4-3 = -4+5 \\ +1 = 1 \quad \checkmark$$

When  $t=1$        $x=2$ ,  $y=-3$  and  $z=1$

point  $(2, -3, 1)$ .  $\checkmark$

(i) direction vectors  $\underline{d}_1 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$  and  $\underline{d}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ,

$$|\underline{d}_1| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$$

$$\underline{d}_1 \cdot \underline{d}_2 = 1 + 3 + 8 = 12.$$

$$|\underline{d}_2| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$

$$\cos\theta = \frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1| |\underline{d}_2|}$$

$$= \frac{12}{\sqrt{26} \sqrt{6}}$$

$$\theta = 16.1^\circ \quad (3 \text{ sig figs})$$

(3)(ii) normal vector perpendicular to  $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 1 & -1 & 2 \end{vmatrix} \\ = (-6 - (-4))\underline{i} - (2 - 4)\underline{j} + (-1 - (-3))\underline{k} \\ = -2\underline{i} + 2\underline{j} + 2\underline{k}$$

$$\underline{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \checkmark \quad (\text{in column form})$$

Plane

$$\underline{n} \cdot \underline{r} = \underline{n} \cdot \underline{a}$$

$$\underline{r} \cdot (\underline{r} - \underline{a}) = \underline{0}$$

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$-x + y + z = -2 - 3 + 1$$

$$x - y - z = 4.$$

④ Prove

$$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$$

n=1

$$\text{LHS} = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

So result is true for  $n=1$

Assume result is true for  $n=k$ .

i.e

$$\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Consider  $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)}$$

We must show that  
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$

$$= \sum_{r=1}^k \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{(k+1)+1}$$

Hence result is true for  $n=k+1$

Since result is true for  $n=1$  and if result holds for  $n=k$  then it follows that result is true for  $n=k+1$  then by induction the result is true for all  $n \in \mathbb{N}$

$$\textcircled{5} \quad 432_5$$

$$\Rightarrow 4 \times 5^2 + 3 \times 5 + 2 = 117.$$

$$117 \div 7 = 16 \text{ remainder } 5$$

$$16 \div 7 = 2 \text{ remainder } 2.$$

$$2 \div 7 = 0 \text{ remainder } 2.$$

$$\text{so } 432_5 = 225_7$$

$$\textcircled{6} \quad f(x) = \ln\left(1 - \frac{x}{2}\right)$$

$$\begin{aligned} f(0) &= \ln 1 \\ &= 0 \end{aligned}$$

$$f'(x) = \frac{1}{1 - \frac{x}{2}} \cdot \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2-x}$$

$$f'(0) = -\frac{1}{2}.$$

$$f''(x) = (2-x)^{-2} \cdot (-1)$$

$$= -\frac{1}{(2-x)^2}$$

$$f''(0) = -\frac{1}{4}.$$

$$f'''(x) = 2(2-x)^{-3} \cdot (-1)$$

$$= -\frac{2}{(2-x)^3}$$

$$f'''(0) = -\frac{1}{4}$$

$$f'''(x) = 6(2-x)^{-4}(-1)$$

$$= -\frac{6}{(2-x)^4}$$

$$f'''(0) = \frac{-6}{16} = -\frac{3}{8}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$$

$$f(x) = 0 + \left(-\frac{1}{2}\right)x + \left(-\frac{1}{4}\right)\frac{x^2}{2} + \left(-\frac{1}{4}\right)\frac{x^3}{6} + \left(-\frac{3}{8}\right)\frac{x^4}{24}$$

$$f(x) = -\frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} + \dots$$

$$\ln\left(1-\frac{x}{2}\right) = -\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} + \dots$$

$$\ln(2-x) = \ln 2 \left(1 - \frac{x}{2}\right)$$

$$= \ln 2 + \ln\left(1 - \frac{x}{2}\right)$$

$$= \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4 + \dots$$

as required.

$$\textcircled{7} \quad (a) \quad f(x) = ax^3 + bx^2 + cx + 1$$

Point  $(1, 0)$  so  $a+b+c+1 = 0.$   
 $\begin{array}{c} x \\ \uparrow \\ y \end{array}$   $a+b+c = -1 \checkmark$

Point  $(-2, 9)$  so  $-8a + 4b - 2c + 1 = 9$   
 $-8a + 4b - 2c - 8 = 0$   
 $4a - 2b + c = -4 \checkmark$

Stationary point  $f'(x) = 3ax^2 + 2bx + c \checkmark$   
when  $x = -2$

so  $f'(-2) = 0$   
 $12a - 4b + c = 0. \checkmark$

(4)

$$(b) \quad \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 4 & -2 & 1 & -4 \\ 12 & -4 & 1 & 0 \end{array} \right) \checkmark$$

$$R2 - 4R1 \quad \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -6 & -3 & 0 \\ 0 & 2 & -2 & 12 \end{array} \right) \checkmark$$

$$R3 - 3R2 \quad \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -6 & -3 & 0 \\ 0 & 0 & -9 & 36 \end{array} \right) \checkmark$$

$$3R3 + R2 \quad \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -6 & -3 & 0 \\ 0 & 0 & -9 & 36 \end{array} \right) \checkmark$$

$$30 - 9c = 36$$

$$c = -4$$

$$-6b - 3c = 0$$

$$-6b + 12 = 0$$

$$b = 2$$

$$a + b + c = -1$$

$$a + 2 - 4 = -1$$

$$a = 1$$

Solusun  $a = 1, b = 2, c = -4$

⑧

$$3A - 2B$$

$$= \begin{pmatrix} 9 & 3 & 3 \\ 0 & 9 & 3\alpha \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 2 & 4 \\ -2 & 8 & 6 \\ 4 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 3\alpha - 6 \\ -1 & -2 & 3 \end{pmatrix}$$

Singular  $\Rightarrow \det(3A - 2B) = 0$

$$1(3 + 2(3\alpha - 6)) - 1(6 + (3\alpha - 6)) - 1(-4 + 1) = 0$$

$$3 + 6\alpha - 12 - 6 - 3\alpha + 6 + 3 = 0$$

$$3\alpha = 6$$

$$\underline{\alpha = 2}$$