

# Cfe AH Maths Mini Prelim

## Practice Paper 1

1.  $P = \begin{pmatrix} 6 & -3 \\ 2 & -5 \end{pmatrix}$ ,  $Q = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$  and  $R = P - 2Q$ .

Find  $R^{-1}$ , the inverse of  $R$ .

3

2. Use Gaussian elimination to solve the system of equations

$$\begin{aligned} 2x - 7y + 10z &= -1 \\ x - 3y + 4z &= 2 \\ 5x - 18y + 26z &= -6 \end{aligned}$$

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3. (a) The line  $l$  has equation  $\frac{x+1}{3} = \frac{y+1}{4} = \frac{z-1}{-2}$ . This line meets the plane  $\pi$  with equation  $2x - y - 4z = 9$  at the point  $T$ .

Find the coordinates of  $T$ .

4

- (b) Find the size of the angle between the line  $l$  and the plane  $\pi$ .

4

- (c) A second plane  $\alpha$  is parallel to the plane  $\pi$  and the line  $l$  meets the plane  $\alpha$  at the point  $R (-5, -9, 5)$ .

Find the equation of the plane  $\alpha$ .

3

4. (a) Find the general solution of the first order linear differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = 2\cos^3 x \sin x - 1, \quad 0 \leq x < \frac{\pi}{2}.$$

8

- (b) Find the particular solution corresponding to the condition that  $y = 3\sqrt{2}$

when  $x = \frac{\pi}{4}$

2

5. Prove by induction that  $\sum_{r=1}^n \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$  for all positive integers  $n$ .

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State the value of the limit as  $n \rightarrow \infty$  of  $\sum_{r=1}^n \frac{3}{(3r-1)(3r+2)}$ .

1

6. Express the integer 271 in base 6.

- (7) (a) Given  $A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix}$  and  $B = \begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$ , find  $AB$ .

1

- (b) Hence solve the system of equations

$$\begin{aligned} 4x + y - 2z &= 1 \\ -x + z &= -2 \\ 2x + y - z &= 5. \end{aligned}$$

3

- (8) Let  $z = \cos \theta + i \sin \theta$ .

- (a) Use de Moivre's theorem to express  $z^3$  in terms of  $3\theta$ .

1

- (b) Use the binomial theorem to express  $z^3$  in terms of  $\sin \theta$  and  $\cos \theta$ .

2

- (c) Hence express

- (i)  $\cos 3\theta$  in terms of  $\cos \theta$

- (ii)  $\sin 3\theta$  in terms of  $\sin \theta$ .

2,2

- (d) Use your answers to (c)(i) and (c)(ii) to show that

$$\cot 3\theta = \frac{1 - 3\tan^2 \theta}{3\tan \theta - \tan^3 \theta}$$

4

# Cfe ALE Maths Mini Prelim. Practice (1)

①  $R = P - 2G$

$$= \begin{pmatrix} 6 & -3 \\ 2 & -5 \end{pmatrix} - 2 \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 \\ 2 & -5 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 4 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} \quad \det R = \frac{4-2}{2} = 2$$

$R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$

②  $\left( \begin{array}{ccc|c} 2 & -7 & 10 & -1 \\ 1 & -3 & 4 & 2 \\ 5 & -18 & 26 & -6 \end{array} \right)$

$2R_2 - R_1 \left( \begin{array}{ccc|c} 2 & -7 & 10 & -1 \\ 0 & 1 & -2 & 5 \\ 5 & -18 & 26 & -6 \end{array} \right)$

$R_3 - 5R_2 \left( \begin{array}{ccc|c} 2 & -7 & 10 & -1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & -1 \end{array} \right)$

Equations are inconsistent  $\Rightarrow$  no solution.

③ (a)  $\frac{x-1}{3} = \frac{y+1}{4} = \frac{z-1}{-2} \quad (= t)$

so  $x = 3t + 1$   
 $y = 4t - 1$   
 $z = -2t + 1$

Substitute into  $2x - y - 4z = 9$

$$2(3t+1) - (4t-1) - 4(-2t+1) = 9$$

$$6t + 2 - 4t + 1 + 8t - 4 = 9$$

$$10t = 10$$

$$t = 1$$

$$\Rightarrow x = 4, y = 3, z = -1 \quad T(4, 3, -1)$$

$$(b) \text{ direction vector of line } \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$$\text{direction vector of plane } \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = 6 - 4 + 8 = 10.$$

$$\cos \theta = \frac{10}{\sqrt{9+16+16} \sqrt{4+1+16}}$$

$$\theta = 66.1^\circ$$

$$\text{angle between line and plane} = 90 - 66.1^\circ = 23.9^\circ$$

(c) Same normal vector so

$$2x - y - 4z = R.$$

Substitute point.

$$-10 + 9 - 20 = R$$

$$R = -21$$

$$\text{plane } 2x - y - 4z = -21.$$

$$(4) (a) e^{\int \frac{\sin x}{\cos x} dx}$$

$$\int \frac{\sin x}{\cos x} = -\ln |\cos x|$$

$$\begin{aligned} & e^{\int \frac{\sin x}{\cos x} dx} \\ &= e^{-\ln |\cos x|} \\ &= e^{\ln |\sec x|} \\ &= \sec x \end{aligned}$$

$$(b) (i) \cos x \frac{dy}{dx} + \sin x y = 2 \cos x \sin x - 1 \quad 0 \leq x < \frac{\pi}{2}$$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = 2 \cos x \sin x - \frac{1}{\cos x}$$

$$\frac{d}{dx} (\sec x y) = 2 \cos x \sin x - \sec^2 x$$

$$\sec x y = \int (\sin 2x - \sec^2 x) dx$$

$$\sec x y = -\frac{1}{2} \cos 2x - \tan x + C.$$

$$y = -\frac{1}{2} \cos x \cos 2x - \sin x + C \cos x.$$

$$(ii) y\left(\frac{\pi}{4}\right) = 3\sqrt{2}.$$

$$\text{when } x = \frac{\pi}{4} \quad y = 3\sqrt{2}$$

$$3\sqrt{2} = -\frac{1}{2} \cos \frac{\pi}{4} \cos \frac{\pi}{2} - \sin \frac{\pi}{4} + C \cos \frac{\pi}{4}$$

$$3\sqrt{2} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} C.$$

$$3\sqrt{2} = -1 + C.$$

$$C = 7$$

$$\text{Particular solution} \quad y = -\frac{1}{2} \cos x \cos 2x - \sin x + 7 \cos x.$$

$$(5) \quad \sum_{r=1}^n \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}.$$

$$\begin{aligned} \text{Let } n=1 \quad \text{LHS} &= \sum_{r=1}^1 \frac{3}{(3r-1)(3r+2)} \\ &= \frac{3}{2 \times 5} \\ &= \frac{3}{10} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{1}{2} - \frac{1}{5} \\ &= \frac{5}{10} - \frac{2}{10} \\ &= \frac{3}{10} \end{aligned}$$

$$\text{LHS} = \text{RHS}.$$

Hence result is true for  $n=1$

Assume result is true for  $n=k$

$$\text{i.e. } \sum_{r=1}^k \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3k+2}$$

Consider  $n = k+1$

$$\begin{aligned} & \sum_{r=1}^{k+1} \frac{3}{(3r-1)(3r+2)} \\ &= \sum_{r=1}^k \frac{3}{(3r-1)(3r+2)} + \frac{3}{(3(k+1)-1)(3(k+1)+2)} \\ &= \frac{1}{2} - \frac{1}{3k+2} + \frac{3}{(3k+2)(3k+5)} \\ &= \frac{1}{2} - \frac{(3k+5)}{(3k+2)(3k+5)} + \frac{3}{(3k+2)(3k+5)} \\ &= \frac{1}{2} - \frac{3k+2}{(3k+2)(3k+5)} \\ &= \frac{1}{2} - \frac{1}{3k+5} \\ &= \frac{1}{2} - \frac{1}{3(k+1)+2} \end{aligned}$$

Hence result is true for  $n=k+1$

Since result is true for  $n=1$  and if result is true for  $n=k$   
then result is true for  $n=k+1$ , by induction the result must  
be true for all  $n \in \mathbb{N}$ .

$$\text{As } n \rightarrow \infty \quad \sum_{r=1}^n \frac{3}{(3r-1)(3r+2)} \rightarrow \frac{1}{2}.$$

$$\lim = \frac{1}{2}$$

$$⑥ \quad 271 \div 6 = 45 \text{ remainder } 1$$

$$45 \div 6 = 7 \text{ remainder } 3$$

$$7 \div 6 = 1 \text{ remainder } 1$$

$$1 \div 6 = 0 \text{ remainder } 1$$

answer 1131,

$$\textcircled{7} \quad (a) \quad AB = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ -8 \end{pmatrix}$$

$$\underline{x = -6, \quad y = 9, \quad z = -8.}$$

$$\textcircled{8} \quad (a) \quad z = \cos\theta + i\sin\theta$$

$$z^3 = (\cos\theta + i\sin\theta)^3$$

$$= (\cos 3\theta + i\sin 3\theta)$$

$$(b) \quad (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta + -3\cos\theta\sin^2\theta - i\sin^3\theta$$

(c) (i) equate real parts.

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

equate imaginary parts.

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta \sin\theta - \sin^3\theta \\&= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\&= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

(Q)  $\cot 3\theta = \frac{\cos 3\theta}{\sin 3\theta}$

$$= \frac{4\cos^3\theta - 3\cos\theta}{3\sin\theta - 4\sin^3\theta}.$$

Divide by  $\cos^3\theta$

$$\cot 3\theta = \frac{4 - 3\sec^4\theta}{3\tan\theta \sec^4\theta - 4\tan^3\theta}$$

Let  $\sec^2\theta = 1 + \tan^2\theta$

$$\cot 3\theta = \frac{4 - 3(1 + \tan^2\theta)}{3\tan\theta(1 + \tan^2\theta) - 4\tan^3\theta}$$

$$= \frac{4 - 3 - 3\tan^2\theta}{3\tan\theta + 3\tan^3\theta - 4\tan^3\theta}$$

$$= \frac{1 - 3\tan^2\theta}{3\tan\theta - \tan^3\theta}$$

as required.