

CfE AH Maths Homework (16)

Questions from AH Paper 2003

① A10. Define $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geq 1$.

(a) Use integration by parts to obtain the value of $I_1 = \int_0^1 x e^{-x} dx$. 3

(b) Similarly, show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$. 4

(c) Evaluate I_3 . 3

② A11. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \quad \text{for } 0 < V < 10.$$

Show that

$$\frac{1}{10} \ln V - \frac{1}{10} \ln (10 - V) = t + C$$

for some constant C . 4

Given that $V(0) = 5$, show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$

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Obtain the limiting value of $V(t)$ as $t \rightarrow \infty$. 2

③ B2. The matrix A is such that $A^2 = 4A - 3I$ where I is the corresponding identity matrix. Find integers p and q such that

$$A^4 = pA + qI.$$

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④ B4. Obtain the Maclaurin series for $f(x) = \sin^2 x$ up to the term in x^4 .
Hence write down a series for $\cos^2 x$ up to the term in x^4 . 4

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⑤ B5. (a) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^n 3(r^2 - r) = (n - 1)n(n + 1).$$

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(b) Hence evaluate $\sum_{r=1}^{10} 3(r^2 - r)$. 2

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⑥ B6. Solve the differential equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^x,$$

given that $y = 2$ and $\frac{dy}{dx} = 1$, when $x = 0$.

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