

BINOMIAL THEOREM

Learning to use
Factorials

$$n! = n(n-1)(n-2) \dots \dots \dots \text{3.2.1} \quad \text{where } n \in \mathbb{N}$$

eg $5!$ (read as 5 factorial)

$$\begin{aligned} &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

note $\boxed{n!}$ button
 ↑
 only accurate for
 small n .
 (upto $10!$)

Note $0! = 1$

Examples

1. Evaluate $\frac{9!}{6!}$ (without the use of a calculator)

$$\begin{aligned} &\frac{9 \times 8 \times 7 \times 6!}{6!} \\ &= 9 \times 8 \times 7 \\ &= 504 \end{aligned}$$

2. Evaluate $8! - 6!$ (without the use of a calculator)

$$\begin{aligned} &8 \times 7 \times 6! - 6! \\ &= 6! (8 \times 7 - 1) \\ &= 55 \times 720 \\ &= 39600 \end{aligned}$$

3. Express in terms of $5!$ (a) $\frac{7!}{7}$ (b) $4!$

$$\begin{aligned} (a) \quad &\frac{7!}{7} \\ &= \frac{7 \times 6 \times 5!}{7} \\ &= 6 \times 5! \end{aligned}$$

$$\begin{aligned} (b) \quad &\frac{4!}{5!} \\ &= \frac{5 \times 4!}{5!} \\ &= \frac{4!}{5} \end{aligned}$$

Advanced Higher Maths : Unit 2
 1.1 Binomial Theorem and Complex Numbers

4. Six children are to be arranged in a line. How many ways can the children be arranged ?

$\underline{6} \quad \underline{5} \quad \underline{4} \quad \underline{3} \quad \underline{2} \quad \underline{1}$

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ = 6!$$

The number of ways of arranging n unlike objects in a line is $n!$.

5. How many ways can 3 out of 5 letters A , B ,C , D , E be arranged ?

$\underline{5} \quad \underline{4} \quad \underline{3}$

$$\text{number of ways} = 5 \times 4 \times 3 \\ = 60$$

6. How many different number plates can be formed if each is to contain the three letters A , C , E followed by 3 digits 4, 7, 8 ?

letters.
 $\underline{3} \quad \underline{2} \quad \underline{1}$

numbers
 $\underline{3} \quad \underline{2} \quad \underline{1}$

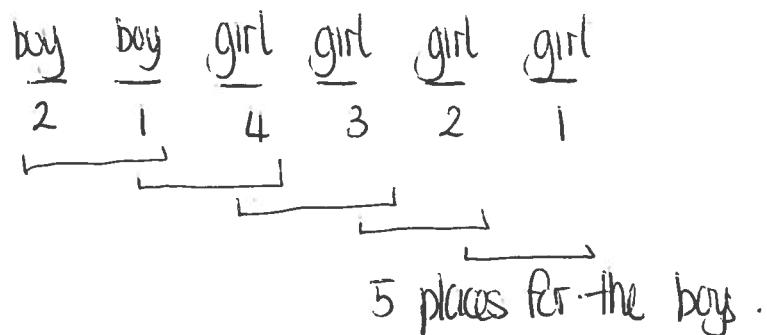
$$\text{number of ways} = 3! \times 3! \\ = 36$$

7. How many even numbers greater than fifty thousand can be made from the numbers 5 , 7 , 2 , 4 , 3 ?

$\begin{matrix} 5 & 7 & 2 & 4 & 3 \end{matrix}$
 ↓ ↓ ↓ ↓ ↓
 5 or 7 3 2 1 2 or 4
 Start with any restriction first

$$2 \times 3 \times 2 \times 2 \times 1 \\ = 24$$

8. Two boys and four girls are to be seated on a bench so that the two boys are next to each other. How many ways can this be done ?



number of ways = $2! \times 4! \times 5$ number of places for boys.
 ↑ ↑
 arrangements of boy. arrangements of girls.
 $= 2 \times 24 \times 5$
 $= 240$

Tips for success

- know the definition for factorial
 $n! = n(n-1)\dots 2 \cdot 1$
- to simplify expressions remember $n!$ can be split up
 eg $n! = n(n-1)!$
 $= n(n-1)(n-2)!$ etc .
- use counting methods to solve problems .

Combinations

*** Order is not important ***

Example

There are 5 people and we have to select a team of 3. How many different teams can we pick ?

If they are ordered in a line

— — —
 5 4 3

$5 \times 4 \times 3 = 60$ ways
 of doing this

If order doesn't matter - possible teams are

team	'rejects'
1 2 3	4 5
1 2 4	3 5
1 2 5	3 4
1 3 4	2 5
1 3 5	2 4
1 4 5	2 3
2 3 4	1 5
2 4 5	1 3
3 4 5	1 2
2 3 5	1 4

Note we have
 an equal number
 of teams and
 rejects

There are 10 possible teams.

i.e. $\frac{5 \times 4 \times 3}{3!}$ ← number of ways of arranging 3 out of 5 in order.

↖ number of ways of arranging the 3 objects
 - counts as one choice since order doesn't matter.

We can write this differently

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{3! \times 2!} = \frac{5!}{3!(5-3)!} = 5C_4 \leftarrow \begin{matrix} \text{'5 choose} \\ 3' \end{matrix}$$

In general the number of different ways of choosing r objects out of n (different) objects is

$${}^n C_r \quad \text{or} \quad \binom{n}{r}$$

where
$${}^n C_r = \frac{n!}{r!(n-r)!}$$

${}^n C_r$
 button
 on
 calculator.

Examples

- Two children are chosen at random from six. How many different ways are there of choosing the children ?

$$\begin{aligned} {}^6 C_2 &= \frac{6!}{2!(6-2)!} \\ &= \frac{6!}{2!4!} \\ &= \frac{6 \times 5}{2!} \\ &= 15 \quad (\text{or use calculator !}) \end{aligned}$$

2. Evaluate $\binom{7}{3}$

$$\begin{aligned} \binom{7}{3} &= {}^7 C_3 \\ &= \frac{7!}{3!4!} \\ &= \frac{7 \times 6 \times 5}{3!} \\ &= 35 \end{aligned}$$

↓
 box marked
 to after
 QL.

Tips for success

- know nC_r means choosing r objects from n (order not important)
- learn the formula ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- use nC_r in problems involving choosing where order is not important.
- be able to use the formula to work out
 nC_r given n and r
 n (or r) given nC_r and r (or n)

3. Find the value of n for which

$$\binom{n}{2} = 55$$

$${}^nC_2 = 55$$

$$\frac{n!}{2!(n-2)!} = 55$$

$$\frac{n(n-1)}{2} = 55$$

$$\frac{n(n-1)}{2} = 110$$

$$n(n-1) = 220$$

$$(n-11)(n+10) = 0$$

$n=11$ or $n=-10$ ← not valid.

$$\text{so } \underline{\underline{n=11}}$$

4. How many different committees each consisting of 3 boys and 2 girls can be chosen from 7 boys and 5 girls ?

number of ways of choosing 3 boys from 7

$$= {}^7C_3$$

$$= 35$$

number of ways of choosing 2 girls from 5

$$= {}^5C_2$$

$$= 10$$

number of committees = 35×10

$$= 350$$

Maths in Action Unit 1 p5 nos 5, 6

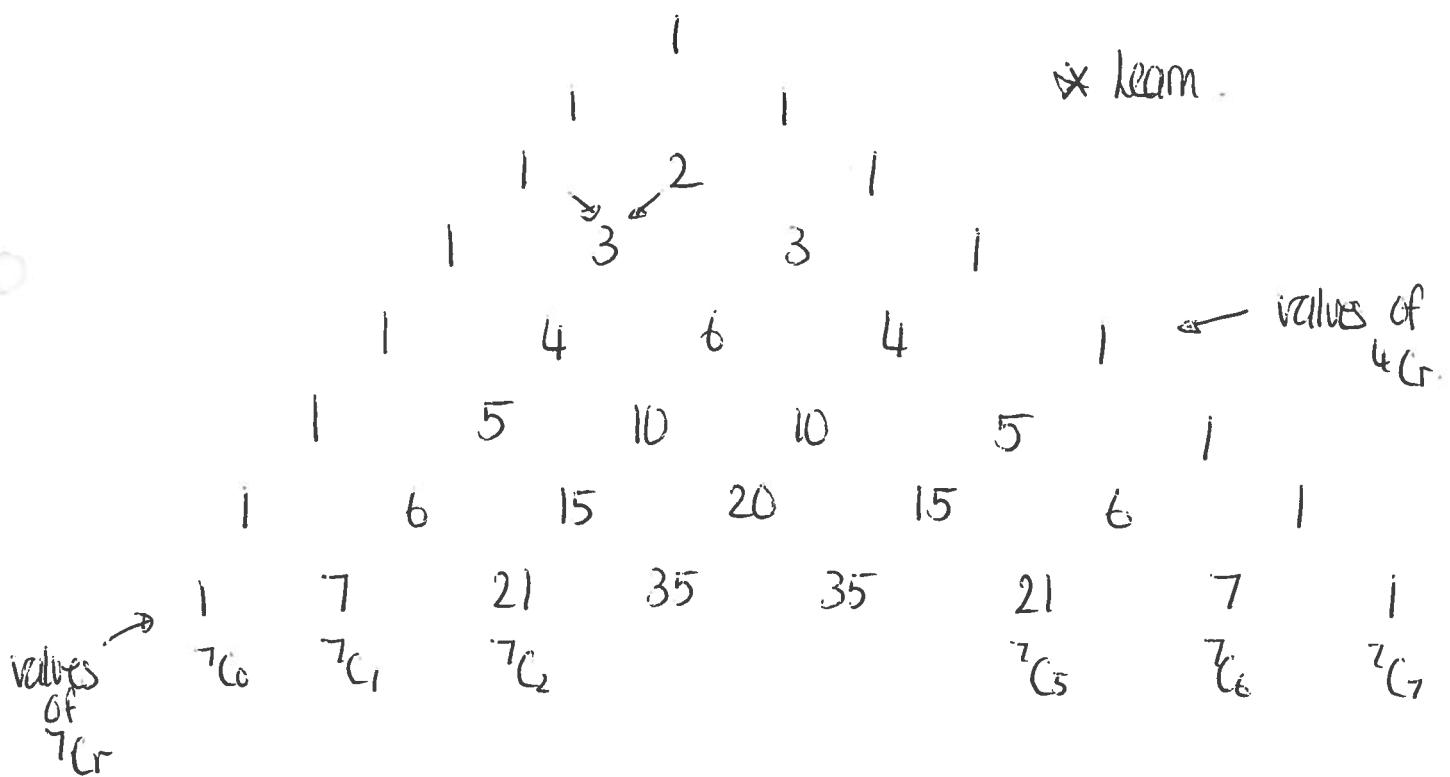
p7 nos 2, 4, 5

Worksheet : Combinations of r objects from n .

Multiply for
 and
 ie boys
 and girls

Pascals Triangle

${}^n C_r$ is the r th number in the n^{th} line of Pascals Triangle



From Pascals triangle we can see two results.

$${}^nC_r = {}^nC_{n-r}$$

(symmetrical)

eg ${}^7C_2 = {}^7C_5$
 add up to 7

$${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$$

(each entry is the sum of the two above)

$$⑨ \quad {}^6C_4 + {}^6C_5 = {}^7C_5$$

We can also prove these results algebraically.

$$\text{Result 1} \quad nHs = {}^nC_r \\ = \frac{n!}{r!(n-r)!}$$

$$\begin{aligned} \text{RHS} &= \frac{n!}{(n-r)! (n-(n-r))!} \\ &= \frac{n!}{(n-r)! r!} \end{aligned}$$

so LHS = RHS Hence $(n-r)! r!$ result.

Result 2 $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$$\text{LHS} = \binom{n}{r-1} + \binom{n}{r}$$

$$= \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!r}{r(r-1)!(n-r+1)!} + \frac{n!(n-r+1)}{r!(n-r)!(n-r+1)!}$$

$$= \frac{n!r}{r!(n-r+1)!} + \frac{(n-r+1)n!}{r!(n-r+1)!}$$

$$= \frac{n!(r+(n-r+1))}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n+1-r)!}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$$= \binom{n+1}{r}$$

= RHS as required.

← make common denominator

Tips for success

- learn Pascal's triangle

$$\begin{array}{ccccccc} & & & 1 & 1 & & \\ & & & 1 & 2 & 1 & \\ & & & 1 & 3 & 3 & 1 \\ & & & 1 & 4 & 6 & 4 \\ & & & & & & \text{etc.} \end{array}$$

- know the result $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

The Binomial Theorem

Expand $(x+y)^0, (x+y)^1, (x+y)^2, (x+y)^3, (x+y)^4$

Can you spot a pattern ?

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

↑ coefficients are the numbers from Pascal's Δ.

1				
	1	2	1	
		1	3	3
			1	4

In pure maths combinations are most often used when expanding brackets.

Eg $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

↑ coefficients from third line in Pascal's Δ.

In general the Binomial Theorem gives

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots$$

$$\dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n x^0 y^n$$



In shorter form this can be written

$$(x+y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$$



LEARN

means the 'sum of'
ie add up each term with $r=0, r=1$ up to
 $r=n$

Note

The coefficients are from Pascals triangle.

The coefficients of the term x^r will be ${}^n C_r$.

The indices of x and y in any term always add up to n.

Examples

1. Expand $(x+y)^5$

$(x+y)^5$ use 5th row of Pascals Δ

1	1	1	1	1
1	2	1	1	1
1	3	3	1	1
1	4	6	4	1
1	5	10	10	5

$$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1xy^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2. Expand $(2x+y)^4$

$$\begin{aligned} (2x+y)^4 &= 1(2x)^4y^0 + 4(2x)^3y^1 + 6(2x)^2y^2 + 4(2x)^1y^3 \\ &\quad + 1(2x)^0y^4 \\ &= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4 \end{aligned}$$

↑ notice that the numbers from Pascals Δ have been obscured.

3. Expand $(x+3)^3$

$$= x^3 + 3x^2 \cdot 3^1 + 3x^1 \cdot 3^2 + 1x^0 3^3$$

$$= x^3 + 9x^2 + 27x + 27$$

4. Expand $(1 - 2x)^3$ treat as $(1 + (-2x))^3$

$$\begin{aligned}
 &= 1 \cdot 1^3(-2x)^0 + 3 \cdot 1^2(-2x)^1 + 3 \cdot 1^1(-2x)^2 + 1 \cdot 1^0(-2x)^3 \\
 &= 1 - 6x + 3(4x^2) + (-2x)^3 \\
 &= 1 - 6x + 12x^2 - 8x^3
 \end{aligned}$$

Notice the pattern of signs + - + -
 This will always be the case when a negative is involved.

5. For $(x+1)^{10}$ expand only as far as the term in x to the power 5.

This means ascending powers of x.

$$\begin{aligned}
 (x+1)^{10} &= (1+x)^{10} \\
 &= {}^{10}C_0 1^{10}x^0 + {}^{10}C_1 1^9x^1 + {}^{10}C_2 1^8x^2 + {}^{10}C_3 1^7x^3 \\
 &\quad + {}^{10}C_4 1^6x^4 + {}^{10}C_5 1^5x^5 + \dots \\
 &= 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + \dots
 \end{aligned}$$

Only expected to know up to 7th row in Pascal's triangle
 → can use nCr instead
 + calculator

Note

It may seem strange to only go up to the term in n=5. This is actually quite common since we often use series expansions like this when we are working with small values of x ($x \approx 0$). In this case the higher powered terms will be of diminishing significance

6. Expand $(x^2 + \frac{1}{x})^3$

$$\begin{aligned}
 &= 1 (x^2)^3(x^{-1})^0 + 3(x^2)^2(x^{-1})^1 + 3(x^2)^1(x^{-1})^2 \\
 &\quad + 1 (x^2)^0(x^{-1})^3
 \end{aligned}$$

$$= x^6 + 3x^4x^{-1} + 3x^2x^{-2} + x^{-3}$$

$$= x^6 + 3x^3 + 3 + \frac{1}{x^3}$$

1 1 1
 1 2 1
 1 3 3 1

7. Expand $(2+x)^5$.

Use this to calculate 2.1^5 .

$$(2+x)^5 = 1 \cdot 2^5 + 5 \cdot 2^4 x^1 + 10 \cdot 2^3 x^2 + 10 \cdot 2^2 x^3 + 5 \cdot 2 x^4 + 1 x^5 \\ = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

$$(2.1)^5 = (2+0.1)^5 \\ = 32 + 80(0.1) + 80(0.1)^2 + 40(0.1)^3 + 10(0.1)^4 + (0.1)^5 \\ = 32 + 8 + 0.8 + 0.04 + 0.001 + 0.00001 \\ = 40.84101$$

Tips for success

When expanding a bracket

- draw Pascal's triangle as far as line matching the power in the expansion.
- expand using coefficients from Pascal's triangle decreasing powers of first term, increasing powers of second.
- take negative sign with term if applicable.
- always Simplify

Maths in Action unit 1 p9 Exercise 3A Questions 1,2

p13 Exercise 4 Questions 1(a),2(a)

Binomial Theorem to find Particular Terms

Further Examples

1. Find the term in x^7 in $(x + \frac{2}{x})^9$

The general term is

$${}^9C_r x^{9-r} \left(\frac{2}{x}\right)^r$$

split up

$$= {}^9C_r 2^r x^{9-r} x^{-r}$$

$$= {}^9C_r 2^r x^{9-2r}$$

get all x's together

For $x^7 \Rightarrow 9-2r=7$
 $r=1$

term ${}^9C_1 2^1 x^{9-2}$

$$= 9 \times 2 x^7$$

$$= 18x^7$$

no x's ie x^0

2. Find the term independent of x in the binomial expansion of

$$(3x - \frac{2}{x^2})^{18}$$

General Term

$${}^{18}C_r (3x)^{18-r} \left(-\frac{2}{x^2}\right)^r$$

$$= {}^{18}C_r 3^{18-r} x^{18-r} (-2)^r \left(\frac{1}{x^2}\right)^r$$

$$= {}^{18}C_r 3^{18-r} (-2)^r x^{18-r} x^{-2r}$$

$$= {}^{18}C_r 3^{18-r} (-2)^r x^{18-3r}$$

Term independent of x when $18-3r=0$

$$r=6$$

$$\text{term } {}^{18}C_6 3^{12} (-2)^6 x^0$$

$$= {}^{18}C_6 3^{12} 2^6$$

3. Find the term independent of x in the expansion of

$$\begin{aligned}
 & (1+x^2)(2x + \frac{1}{x})^{10} \\
 &= (2x + \frac{1}{x})^{10} + x^2 (2x + \frac{1}{x})^{10} \quad \rightarrow \text{ multiply out first bracket} \\
 &\qquad\qquad\qquad \curvearrowleft \text{General term} \qquad\qquad\qquad \curvearrowright \text{general term} \\
 &\qquad\qquad\qquad {}^{10}C_r (2x)^{10-r} \left(\frac{1}{x}\right)^r \\
 &= {}^{10}C_r 2^{10-r} x^{10-r} x^{-r} \\
 &= {}^{10}C_r 2^{10-r} x^{10-2r} \\
 &\qquad\qquad\qquad \text{independent of } x \quad 10-2r=0 \\
 &\qquad\qquad\qquad r=5 \\
 &\text{term } {}^{10}C_5 2^5 x^0 \\
 &= 8064 \quad \qquad\qquad\qquad \text{so } {}^{10}C_6 2^4 x^0 = 3360 \\
 &\qquad\qquad\qquad \text{total } 8064 + 3360 \\
 &\qquad\qquad\qquad = 11424
 \end{aligned}$$

Tips for success

- remember the general term for $(x+y)^n$ ie ${}^nC_r x^{n-r} y^r$
- replace value of n and appropriate terms for x and y
- split up terms as far as possible .
- gather x 's together
- set power of x to appropriate value
- solve to get r and substitute to get term .

COMPLEX NUMBERS

To enable us to solve equations such as

$$x^2 = -16$$

we introduce the idea of an imaginary number i where

$$i = \sqrt{-1} \quad \text{or} \quad i^2 = -1$$

Thus we have

$$\begin{aligned} x^2 &= -16 \\ x &= \pm\sqrt{-16} \\ x &= \pm\sqrt{16}\sqrt{-1} \\ x &= \pm 4i \end{aligned}$$

Definitions

- bi where $b \in R$ is an imaginary number.
- $a + bi$ where $a, b \in R$ is a complex number.
- If $z = a + ib$ then the real part of z is $\text{Re}(z) = a$ and the imaginary part of z is $\text{Im}(z) = b$

eg If $z = 2 - 5i$ we have $\text{Re}(z) = 2$

and $\text{Im}(z) = -5$

Note If two complex numbers are equal then their real parts are equal and their imaginary parts are equal.

eg $6 + 4i = a + ib$

so $a = 6$

and $b = 4$

- The **complex conjugate** of $z = a + bi$ is $\bar{z} = a - bi$
- eg Write down the complex conjugate of the following complex numbers

(a) $z = -4 + 2i$

$$\bar{z} = -4 - 2i$$

(b) $w = 3 - 5i$

$$\bar{w} = 3 + 5i$$

Adding and Subtracting

Simply add or subtract the real parts together and the imaginary parts together.

Examples

1. Simplify $(4 - 2i) + (3 + 7i)$

$$= 7 + 5i$$

2. Find $z + \bar{z}$ if $z = 3 - 7i$

$$\begin{aligned}
 & z + \bar{z} \\
 & = 3 - 7i + 3 + 7i \\
 & = 6
 \end{aligned}$$

Multiplication

Multiply brackets as usual.

Remember

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

etc

Examples

1. Calculate $(2 - 7i)(3 + 2i)$

$$\begin{aligned} &= 6 + 4i - 21i - 14i^2 \\ &= 6 - 17i + 14 \\ &= 20 - 17i \end{aligned}$$

2. If $z = a + ib$ calculate $z\bar{z}$.

$$\begin{aligned} z\bar{z} &= (a+ib)(a-ib) \\ &= a^2 - i^2 b^2 \\ &= a^2 + b^2 \end{aligned}$$

Note $z\bar{z}$ is always real $\times \times$

Maths in Action Book 2 Page 90 Exercise 1 Question 1, 7, 8

Division

To simplify $\frac{a+ib}{c+id}$ multiply top and bottom by the complex conjugate of the denominator. i.e. $c - id$

Example

Simplify $\frac{5+2i}{1-3i}$

$$\begin{aligned} &= \frac{(5+2i)(1+3i)}{(1-3i)(1+3i)} \\ &= \frac{5+15i+2i+6i^2}{1-9i^2} \\ &= \frac{5+17i-6}{1+9} \\ &= \frac{-1+17i}{10} \\ &= -\frac{1}{10} + \frac{17}{10}i \end{aligned}$$

Exam Question

Two complex numbers, z_1 and z_2 , are given by $z_1 = 3 - 2i$ and $z_2 = 6 + ki$, where k is a real number.

- (a) Given that $z_1^2 - 3z_2$ is a purely real number, find the value of k .

- (b) Express $\frac{\bar{z}_2}{z_1}$ in the form $x + iy$, where x and y are real numbers.

$$\begin{aligned}
 (a) \quad & z_1^2 - 3z_2 \\
 &= (3-2i)^2 - 3(6+ki) \\
 &= 9 - 12i + 4i^2 - 18 - 3ki \\
 &= 9 - 12i - 4 - 18 - 3ki \\
 &= -13 - 12i - 3ki \\
 &= -13 - i(12+3k)
 \end{aligned}$$

$$\begin{aligned}
 & \text{real so } 12+3k=0 \\
 & k=-4
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \frac{\bar{z}_2}{z_1} &= \frac{6+4i}{3-2i} \\
 &= \frac{(6+4i)(3+2i)}{(3-2i)(3+2i)} \\
 &= \frac{18+12i+12i+8i^2}{9-4i^2} \\
 &= \frac{10+24i}{13} \\
 &= \frac{10}{13} + \frac{24}{13}i
 \end{aligned}$$

Tips for success.....

- remember $i^2 = -1$, $i^3 = -i$ etc .
- $z\bar{z}$ is always real so to divide multiply bottom by complex conjugate .
- purely real means coefficient of i is zero .
- purely imaginary means real part is zero .

Maths in Action Book 2 Page 91 Exercise 2 Question 1 (a) (d) , 3 , 6

Equations

We can now solve all quadratic equations.

Example

① Solve $x^2 + 2x + 6 = 0$

$$x \in \mathbb{C}$$

↪ x can be a complex number .

Here we have real coefficient \Rightarrow use quadratic formula .

$$a=1 \quad b=2 \quad c=6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = -2 \pm \sqrt{4 - 24}$$

$$x = \frac{-2 \pm \sqrt{-20}}{2}$$

$$x = \frac{-2 \pm i\sqrt{20}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{5}i}{2}$$

$$x = -1 \pm \sqrt{5}i$$

* type up by
 (2) Solve $z^2 + 2iz - 6 = 0 \quad z \in \mathbb{C}$

Let $z = a+bi \quad a, b \in \mathbb{R}$

$$a^2 + 2abi - b^2 + 2ai - 2b - 6 = 0$$

Equate
(real)

$$a^2 - b^2 - 2b - 6 = 0 \quad \dots (1)$$

(imaginary)

$$2ab + 2a = 0$$

$$2a(b+1) = 0$$

$$a=0 \text{ or } b=-1$$

$$a=0$$

$$\text{In (1)} \quad -b^2 - 2b - 6 = 0$$

$$b^2 + 2b + 6 = 0$$

$$b^2 - 4ac = 4 - 24$$

$$= -20$$

no real solutions.

Imaginary coefficients
easier to let $z = a+bi$
and equate real
and imaginary
parts

$$b = -1$$

$$\text{In (1)} \quad a^2 - 1 + 2 - 6 = 0$$

$$a^2 = 5$$

$$a = \pm\sqrt{5}$$

Solutions $z = \sqrt{5} - i \text{ or } -\sqrt{5} - i$

Finding the Square Roots of a Complex Number

Example

Find the square roots of $(3 + 4i)$

Suppose the square root is $a+bi$ where $a, b \in \mathbb{R}$.

$$\begin{aligned} \text{then } (3+4i) &= (a+bi)^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

Equate real and imaginary parts.

$$\text{so } a^2 - b^2 = 3 \dots (1) \quad (\text{real parts})$$

$$2ab = 4 \dots (2).$$

Solve simultaneously

PTO.

$$\text{From } ② \quad a = \frac{2}{b}$$

$$\text{In } ① \quad \left(\frac{2}{b}\right)^2 - b^2 = 3$$

$$\frac{4}{b^2} - b^2 = 3$$

$$4 - b^4 = 3b^2$$

$$b^4 + 3b^2 - 4 = 0$$

$$(b^2 + 4)(b^2 - 1) = 0$$

$$b^2 = -4$$

†

not possible
since $b \in \mathbb{R}$

$$b^2 = 1$$

$$b = \pm 1$$

$$b = 1 \Rightarrow a = 2$$

$$b = -1 \Rightarrow a = -2$$

Square roots are $2+i$
 $-2-i$

Tips for success.....

- Solving quadratic with real coefficients \Rightarrow use quadratic formula.
- Solving quadratic with imaginary or complex coefficients
 \rightarrow let $z = a+ib$ and equate real and imaginary parts.
- Finding square root of a complex number z
 Let $z = (a+ib)^2$ and equate real and imaginary parts.

Solving Polynomials

Example

Solve $x^3 - 5x^2 + 9x - 5 = 0 \quad x \in \mathbb{C}$.

We must find a factor first \rightarrow trial and error.

Try 1 | 1 -5 9 -5

	1	-5	9	-5
	1	-4	5	
	1	-4	5	0

remainder 0
so $x-1$ is a factor

$$x^3 - 5x^2 + 9x - 5 = 0$$

$$(x-1)(x^2 - 4x + 5) = 0$$

$$\text{so } x-1=0$$

$$x=1$$

or

$$x^2 - 4x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-20}}{2}$$

$$= \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

doesn't factorise

Solutions $\underline{x=1, 2+i, 2-i}$

For a polynomial with real coefficients if $a+ib$ is a root

then so is $a-ib$

Any polynomial of order n has n roots.

Example

Given that $z = 1 - i$ is a root of the polynomial equation $z^4 + 4z^3 - 8z + 20 = 0$
find the other roots.

Polynomial has real coefficients.

so $z = 1 - i$ is a root $\Rightarrow z = 1 + i$ is also a root.

Factors $(z - (1-i))(z - (1+i)) \rightarrow$ multiply out

$$\begin{aligned} &= z^2 - z(1+i) - z(1-i) + (1-i)(1+i) \\ &= z^2 - z - iz - z + iz + 1 - i + i + 1 \\ &= z^2 - 2z + 2. \end{aligned}$$

↑ since $-i^2 = 1$

Divide into polynomial

$$\begin{array}{r} z^2 + 6z + 10 \\ \hline z^2 - 2z + 2 \quad | \quad z^4 + 4z^3 \quad -8z + 20 \\ z^4 - 2z^3 + 2z^2 \\ \hline 6z^3 - 2z^2 - 8z + 20 \\ 6z^3 - 12z^2 + 12z \\ \hline 10z^2 - 20z + 20 \\ 10z^2 - 20z + 20 \\ \hline 0 \end{array}$$

must go in
exactly since dividing
by factors.

So to get other factors find roots of:

$$\begin{aligned} z^2 + 6z + 10 &= 0 \\ z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ &= \frac{-6 \pm 2i}{2} \\ &= -3 \pm i \end{aligned}$$

Roots are

$$\begin{aligned} &1 \pm i \\ &-3 \pm i \end{aligned}$$