ODD + ODD = ODD IS IT POSSIBLE?

Ilya Sinitsky, Rina Zazkis, and Roza Leikin explore odd and even functions.

This article was inspired by the following observation by a student:

> ***When we started to deal with trigonometric functions, the teacher mentioned that some of them are odd, or even. For example, the function*** $y = \sin x$ ***is an odd function and*** $y = \cos x$ ***is an even function. But what is the reason for using the same adjectives 'odd' and 'even' both for numbers and for functions? It is simply confusing! These are really odd names for functions!***

We decided to explore the reasons why some functions are referred to as odd or even, and the relationships between these functions and odd and even numbers. We start by recalling the definitions – see, for example, Larson & Hostetler, 2006:

> Function $f(x)$ of real argument x is **even** if, and only if, the domain of the function is symmetrical around 0 and the equality $f(-x) = f(x)$ holds for every x from the domain.
> Function $f(x)$ of real argument x is **odd** if, and only if, the domain of the function is symmetrical around 0 and the equality $f(-x) = -f(x)$ holds for every x from the domain.

Simply stated, the value of an even function is the same for a number and its opposite, whereas the value of an odd function changes for the opposite number when the argument is replaced by its opposite. Of course, this requires that for each number x its opposite is also in the domain of the function. According to the definitions, functions of both types are defined at symmetrical parts of the number line – or, for the whole domain of real numbers, $-\infty < x < \infty$.

Graphs of both odd, and even, functions are symmetrical. The graphs of odd functions have a rotational symmetry of 180 degrees with the centre at the origin of axes – also referred to as central symmetry at the point of origin – since point $(-a, -b)$ is on the graph if, and only if, (a, b) point is on the graph of the function. The y-axis is the symmetry line of the graph of every even function, since point $(-a, b)$ is on the graph if, and only if, point (a, b) is on the graph of the function.

However, what reason is there for giving the name 'even' to the function $y = \cos x$ (*Figure 1*), or for the function $y = \sqrt{1-x^2}$ (*Figure 2*)? Perhaps this is due to seeing the graph of an even function as a union of two congruent – up to mirror reflection – parts, as analogous to seeing each even number as the sum of two identical natural numbers?

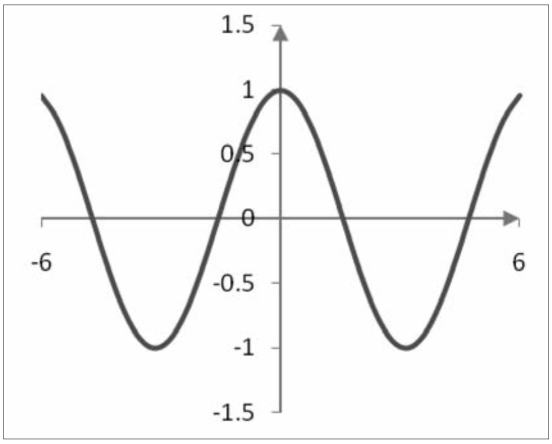


Figure 1: y = cos x is an even function

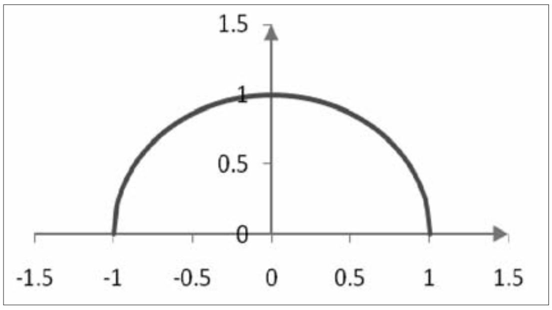


Figure 2: $y = \sqrt{1 - x^2}$ *is an even function*

However, the graphs of odd functions may also be considered as a union of two congruent parts whose values are opposite for opposite arguments. For example, function $y = \sin x$ is an odd function (see *Figure 3*). This view of the odd function does not provide an analogy with odd numbers. Thus, we turn to well-known odd and even functions in which relationships between the functions and the numbers can be clearly seen.

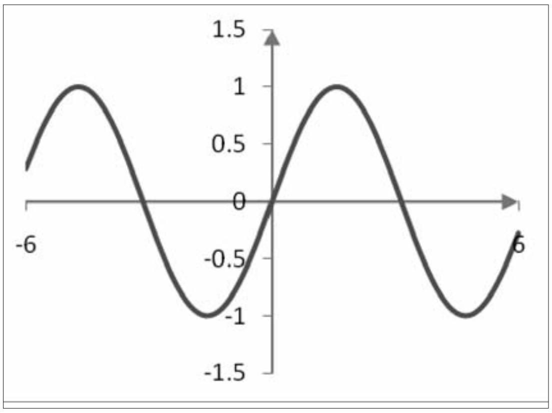


Figure 3: $y = \sin x$ is an odd function

Functions $f(x) = x^n$ for natural values of n.

It is evident that function $f(x) = x^n$ for **every even** natural number ($n=2m$) is even, since it holds the equality: $f(-x) = f(x)$.

$$f(-x) = (-x)^n = (-x)^{2m} = (-1)^{2m} \cdot x^{2m} = x^n = f(x)$$

Among the examples of even functions, one can easily see the quadratic function, $y = x^2$, or the function $y = x^4$ whose graphs – parabolas of different degrees – are symmetrical with respect to the y-axis.

For **every odd** natural number $n=2m+1$, the function $f(x) = x^n$ is an odd function: $f(-x) = -f(x)$.

$$f(-x) = (-x)^n = (-x)^{2m+1} = (-1)^{2m+1} \cdot x^{2m+1} = -x^n = -f(x)$$

In particular, the simplest linear function $y = x$ – where the exponent of the variable is equal to 1 – and $y = x^3$ are odd functions whose graphs are symmetrical with respect to the origin of the coordinate system.

Since the notion of even and odd numbers can be expanded from the natural numbers to integers, this leads us to consider integer exponents. For $n=0$, the function $f(x) = x^n$ becomes constant: $f(x) = x^0 = 1$, and it is an even function in the sense of symmetry with respect to the y-axis.

In the next step we focus on functions $f(x) = x^{-n}$ with natural values of n, and take into account the well-known definition: $x^{-n} = \frac{1}{x^n}$. The behaviour of these functions with negative integer values of exponents differs significantly from the polynomial functions with natural exponents. For example, they are not bounded in the neighbourhood of the point $x=0$, and are not defined at this point. Nevertheless, the symmetry of these functions fits their names: functions $f(x) = x^{-n}$ have a y-axis symmetry for even values of n, and central symmetry for odd exponents, even though the centre of symmetry (0;0) does not belong to the graph of the function – see *Figure 4*. The last case also reminds us that the origin of the coordinate system does not necessarily belong to the graph of every odd function.

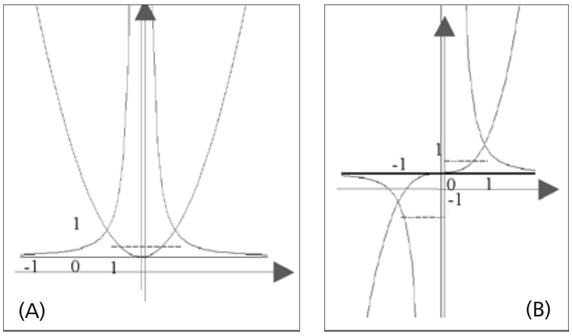


Figure 4: Graphs of functions $y = x^n$ and $y = \frac{1}{x^n}$ for: (A) even values of ($n > 0$); *(B) odd values of* n ($n > 1$)

As can be derived from these examples, the use of the tags 'even' and 'odd' for functions comes as an extension of well-justified terms for even or odd exponents. Since all the functions of the form $f(x) = x^n$ with even n have symmetry with respect to y-axis, it is a logical extension to name all the functions with this symmetry as 'even' ones – and a similar analogy can explain the use of the term 'odd' for functions with central symmetry.

Furthermore, each linear combination of even exponents of a variable, such as $y = x^4 - 3x^2 + 5$ (*Figure 5*) or $y = 3.2x^{12} + 11x^{10} + x^2$, provides an additional example of an even function – and the

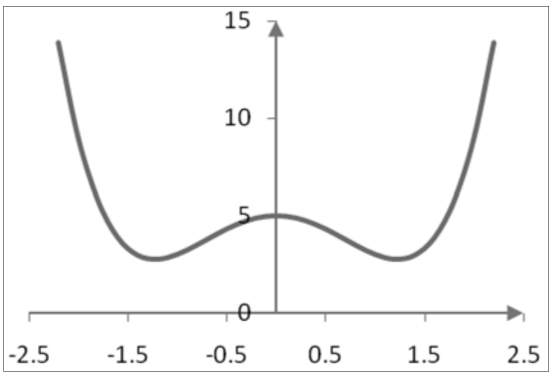


Figure 5: $y = x^4 - 3x^2 + 5$ is an even function

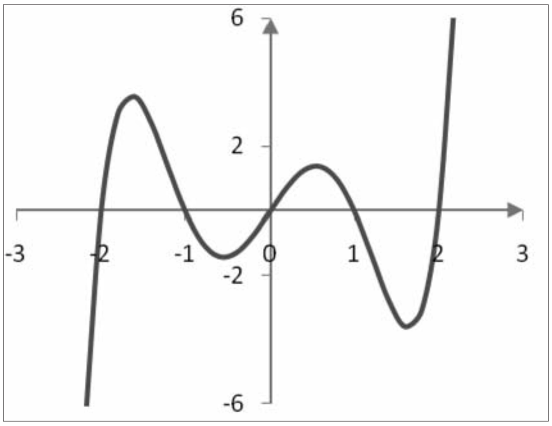


Figure 6: $y = x^5 - 5x^3 + 4x$ is an odd function.

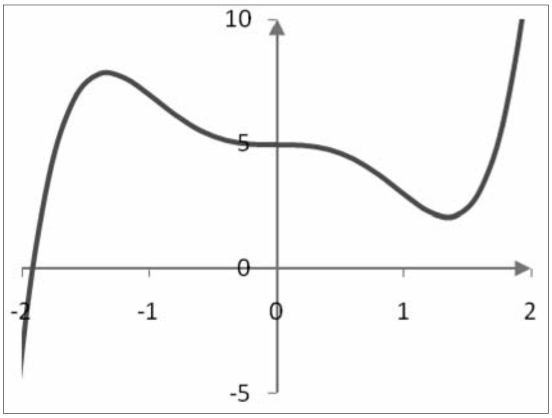


Figure 7: $y = x^5 - 3x^3 + 5$ is neither an odd nor an even function.

analogy is complete for any linear combinations of odd exponents, such as $y = x^5 - 5x^3 + 4x$ (*Figure* 6) or $y = 3.2x^{11} + 11x^7 - x^3$, but *not* for $y = x^5 - 3x^3 + 5$ (*Figure* 7), which can be a possible assessment task for students.

While extending the terminology based on symmetry appears a plausible, logical explanation for the choice of even and odd descriptors for functions, additional reasons can be found by considering more advanced mathematics. We turn to those in the next section.

Considering Maclaurin series

An arbitrary infinitely differentiable function can be represented with Maclaurin – power – series. For details see for example, www.intmath.com/Series-expansion/2_Maclaurin-series.php.

The Maclaurin series is a sum of non-negative integer powers of a variable with coefficients which are determined by the derivates of the function at the point $x=0$:

$$f(x) \sim f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Generally, this series approximates the function in some interval near the point $x=0$.

- What does this series look like for an even function?
- What is the value of its derivative $f'(x)$ for $x=0$?

According to the definition,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

but if this value does exist, for an even function it is necessarily equal to its opposite value:

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(-\Delta x) - f(0)}{-\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{-\Delta x}$$

$$= -\lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = -f'(0)$$

It means that $f'(0) = 0$. Moreover, a very similar argument leads to the equality $f'(x) = -f'(x)$ for every point of differentiability of the initial function. Thus, the derivative of an even function is an odd function.

- What about the derivative of any odd function: $g(x)$?

Supposing $g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = A$

we can derive

$$g'(-x) = \lim_{\Delta x \to 0} \frac{g(-x + \Delta x) - g(-x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-g(x - \Delta x) + g(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-(g(x - \Delta x) - g(x))}{-\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{g(x - \Delta x) - g(x)}{-\Delta x} = A$$

– because the last ratio is just the definition of $g'(x)$, since $(-\Delta x) \to 0$ is equivalent to $\Delta x \to 0$. Thus, the derivative of an arbitrary differentiable odd function is an even one. Now we turn back to an even function: $f(x)$.

The first derivative of an even function is an odd function. What about its second derivative? The second derivative is a derivative of the first derivative function, but the function $f'(x)$ is an odd function, hence its derivative, i.e. $(f'(x))' = f''(x)$ is an even function. This chain of arguments can be drawn further; thus derivatives of odd order ($f'(x)$, $f'''(x), f^{(v)}(x)$, ... of differentiable even functions are odd functions, and derivatives of each even order ($f''(x), f^{(iv)}(x)$, ...) for differentiable even functions are even functions. It means that for $x=0$

$$f'(0) = f'''(0) = f^{(v)}(0) = \ldots = f^{(2n+1)}(0) = 0$$

In other words, all odd exponents of Maclaurin decomposition of an even function have zero coefficients and, in fact, this series consists of even powers of the variable only. Correspondingly, the second, the fourth, the sixth, and every even-order derivatives for odd functions are odd functions with

$f''(0) = f^{(iv)}(x) = \dots = f^{(2n)}(x) = 0$

Therefore, the Maclaurin decomposition of odd functions consists of odd powers of the variable only.

Beyond calculus

Historically, continuous and differentiable functions have been treated within Calculus, and – as shown above – the reason for using the terms 'even function' and 'odd function' are far beyond the formal extension of names of polynomial functions. Nevertheless, we apply those names also to all functions that fit the formal abovementioned definition. In the family of even functions we can see the familiar absolute value function $f(x) = |x|$ that is not differentiable at $x=0$. In the family of odd functions we notice the function –

$$sgn(x) = \begin{cases} -1, \text{for } x<0 \\ 0, \text{for } x=0, \\ 1, \text{for } x>0 \end{cases}$$ which is non-continuous at $x=0$.

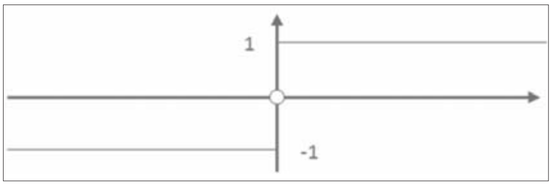


Figure 8: $y = sgn(x)$ is an odd function.

Moreover, the Dirichlet function $\theta(x)$ – defined as $\theta(x) = 1$ for each rational x and $\theta(x) = 0$ for each irrational x – provides an example of an even function that cannot be plotted, but holds the y-axis symmetry – since the rationality or irrationality of a number x is the same as that of its opposite $(-x)$.

Considering arithmetic operations

Is there an analogy between odd/even numbers and odd/even functions with respect to arithmetic operations? If we start with addition and subtraction of even functions, the analogy appears to hold: the sum, and difference, of even functions is an even function with full correspondence to the addition or subtraction of even numbers. However, the sum and difference of odd functions is an odd function – contrary to the expectation by analogy with the set of integers. Extending the investigation to the multiplication and division of functions can be an appropriate task for students using, for example, graphing technology. The results of addition, subtraction, multiplication and division, where possible, of functions are summarised in the following table.

$+,-$	Even function	Odd function
Even function	Even function	
Odd function		Odd function

$\times, \div$	Even function	Odd function
Even function	Even function	Odd function
Odd function	Odd function	Even function

Figure 9: Outcomes of arithmetic operations with odd and even functions.

As seen in *Figure 9*, arithmetic operations with even and odd functions are not in accord with operations with even and odd numbers. However, the most surprising result is the pair of empty cells. We focus on this result in the next section.

Limits of analogy

What is the meaning of the empty cells in the addition/subtraction table? When an odd function is added to an even one – both non-zero, what can be said about the result? Surprisingly or not, the result is neither odd nor even. And this is the main difference between even and odd integers and even and odd functions. Each integer is either even or odd, and the set of integers is the union of two subsets with an empty intersection. This is totally different with functions. First of all, a function exists which is both even and odd. For such a function the following should hold for every x in the domain:

$f(-x) = f(x)$ and $f(-x) = -f(x)$ means $f(x) = -f(x)$

This is possible only for $f(x) = 0$ for any symmetric domain.

More importantly, though, most functions are neither odd nor even. Consider the simple example of the sum of the function $f(x) = x$ and the constant function $g(x) = 1$. The graph of the sum of these functions $y = f(x) + g(x) = x + 1$ is not symmetric, neither with respect to the y-axi, nor with respect to the origin of the axes.

Striving for symmetry

Symmetry properties of even and odd functions are very attractive. Is it possible to use an arbitrary function to construct an even or an odd function? The geometric interpretation of oddness and evenness can assist us in achieving this goal. When function $f(x)$ is even, the equality $f(x) = f(-x)$ holds for every point x from the domain of the function. If the domain of the function is a symmetric one – it includes for each point, x_0 and its opposite, $-x_0$, this function can be modified to be even. The 'non-evenness' of any function for each argument x is caused by the non-zero difference $\Delta(x) = f(x) - f(-x)$.

To achieve equality $\Delta(x) = 0$ we define the value of the new function at this point as an arithmetical mean of the values, $E_f(x) = \frac{1}{2}(f(x) + f(-x))$ (see *Figure 10*). In fact, it is easy to see that for each function $f(x)$, $E_f(x)$ is an even function: $E_f(-x) = \frac{1}{2}(f(-x) + f(-(-x))) = \frac{1}{2}(f(-x) + f(x)) = E_f(x)$.

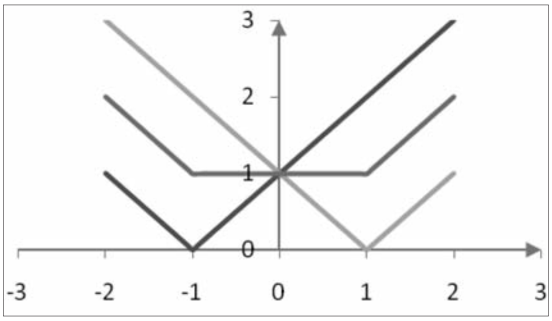
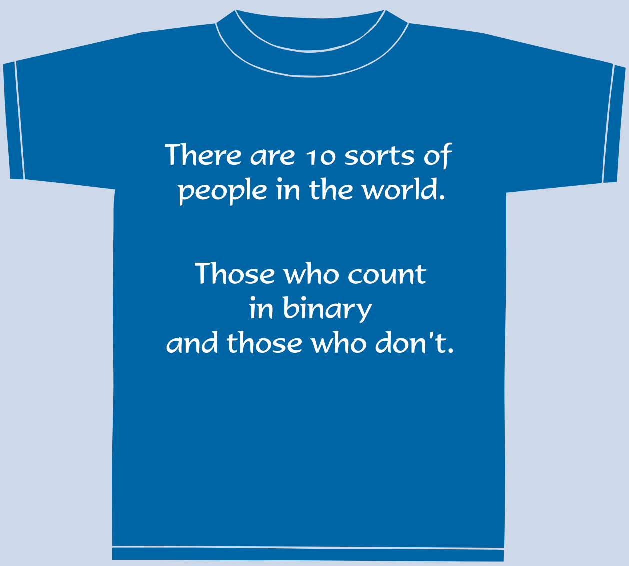


Figure 10: Constructing $E_f(x) = \frac{1}{2}(f(x) + f(-x))$ for $f(x) = |x+1|$

It makes you think . . .

George Knight pauses for thought twice . . .

1 This T-shirt legend was seen in Regent Street, London, in the summer of 2011.



2 Alfie – aged five years and two months – responds to a comment from his granny by saying: "That's not a few, that's five – three is a few!"

There might just be something here worth pondering for say, just a *few moments* . . .

By very similar arguments, for each function with a symmetric domain an odd function can be constructed: $0_f(x) = \frac{1}{2}(f(x) - f(-x))$.

For 'pure' even or odd function these new functions hold special properties:

When function $f(x)$ is even, $E_f(x) \equiv f(x)$ and $0_f(x) \equiv 0$.

When function $f(x)$ is odd, $0_f(x) \equiv f(x)$ and $E_f(x) \equiv 0$.

On the other hand, an arbitrary function $f(x)$ with a symmetric domain may be presented as a sum of even and odd components of this function: $E_f(x) + 0_f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) = f(x)$

Some analogy after all

We have considered arithmetic operations on even and odd functions. However, one important operation on functions so far has not been mentioned. This operation is a composition of functions; that is, an application of one function to the result of another: $g \circ f(x) = g(f(x))$ with an appropriate choice of domain for function g. Exploring the composition of even and odd functions – which is an appropriate task for students with or without graphing technology – leads to the conjecture that the composition of functions upholds the analogy with multiplication of numbers.

The composition is an odd function when all the components of this composition are odd functions:
$g \circ f((-x)) = g(f(-x)) = g(-f(x)) = -g(f(x)) = -g \circ f(x)$.

For example, both $y = \sin^3 x$ and $y = \sin(x^3)$ are odd functions.

The composition of those functions – two or more – is an even function when at least one of the functions is an even one. Here we are happy to leave the proof to the reader. For instance, all of these functions are even:

$y = \sin^2 x = (\sin x)^2$; $y = (\cos x - 1)^3$;
$y = |x^3|$; $y = (x^2 + 1)^5$.

In summary, when you were convinced that *odd + odd = even*, you were obviously thinking about odd numbers. With odd functions it is, without doubt, a different story.

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Reference

Larson, R. & Hostetler, R.P. (2006). *Precalculus*. 7th edition. Brooks Cole.