SEQUENCES AND SERIES

- A **sequence** is an ordered list of numbers where each term is obtained according to a fixed rule.

  $U_1\ U_2\ U_3\ U_4$

  e.g. 1 , 4 , 9 , 16 , ... $U_i = i^2$, $i \in N$

  TERMS

- A **series** is written as $a_1 + a_2 + a_3 + ...$, the terms of which form a sequence $S_n = \sum_{i=1}^{n} a_i$

  e.g 1 + 4 + 9 + 16 + ... $S_n = \sum_{i=1}^{n} i^2$

  $= 1^2 + 2^2 + 3^2 + \dots + n^2$

- The ***nth* term** of a sequence or series is denoted by u_n

  (or a_n)

- A sequence or series can be defined by a recurrence relation where u_{n+1} is given as a function of lower terms.

- A first-order recurrence relation is where $u_{n+1} = f(u_n)$, recall Higher $u_{n+1} = 0.8u_n + 15$

1) Find the first 3 terms of each sequence and state whether it is increasing decreasing or neither

a) $U_i = -2i$ (b) $U_i = 2i + 3$ (c) $U_i = 1 + (-1)^i$

2) Write the following series in expanded form

a) $\sum_{i=1}^{6} 3i$ (b) $\sum_{i=1}^{5} (-1)^i \left(\frac{3}{i}\right)$

Arithmetic Sequences and Series

An **arithmetic sequence** or series is one in which

$$u_{n+1} - u_n = u_n - u_{n-1} \qquad \text{for all values of } n$$

i.e there is a common difference, ***d***, between neighbouring terms in the sequence.

This is the defining property of an arithmetic sequence.

e.g. for the sequence 1, 4 , 7, 10, 13,...... the common difference is 3 and the sequence is defined as **the arithmetic sequence with first term 1 and common difference 3.**

For an **Arithmetic Sequence or Series** with first term ***a*** and common difference ***d***, the ***nth*** term is given by

$$u_n = a + (n-1)d$$

giving $U_1 = a$ $\quad U_4 = a + 3d$
$U_2 = a + d$
$U_3 = a + 2d$

Examples

1) Find an expression for u_n and evaluate u_{11} for the sequence 3, 7, 11, 15,........

$a_1 = 3$ $\quad d = 4$

Note

$a_1 = 3$

$a_2 = 3 + 1(4) = 7$

$a_3 = 3 + 2(4) = 11$

$a_4 = 3 + 3(4) = 15$

ie. $a_n = a_1 + (n-1)d$

here $a_n = 3 + (n-1)4$

$= 4n - 1$

$a_{11} = 4(11) - 1$

$= 43$

using formula,

$a = 3$ $\quad d = 4$

$U_n = a + (n-1)d$

$= 3 + (n-1)\cdot 4$

$= 4n - 1$

$U_{11} = 4(11) - 1$

$= 43$

2) In an arithmetic sequence $u_5 = 23$ and $u_{12} = 37$. Find an expression for the *nth* term of this sequence.

Arithmetic $\Rightarrow$ $U_n = a + (n-1)d$

$\Rightarrow$ $U_5 = a + 4d = 23$

$U_{12} = a + 11d = 37$ (subtracting)

$-7d = -14$

$d = 2$

Given $a + 4d = 23$

$a + 4(2) = 23$

$a = 23 - 8$

$= 15$

Hence $U_n = 15 + (n-1) \cdot 2$

$= 2n + 13$

[check that $U_{12} = 37$!]

3) Given the arithmetic sequence 5, 11, 17, 23,……find the value of n for which $u_n = 113$

$a = 5$ $\quad d = 17 - 11$

$= 6$

Now $U_n = a + (n-1)d = 113$

$5 + (n-1) \cdot 6 = 113$

$5 + 6n - 6 = 113$

$-1 + 6n = 113$

$6n = 114$

$n = 19$

Maths in Action Book 2 Page 117 Exercise 2A Questions 1(a),(d),(g), 2(a),(d),(g) 3 , 4 , 5 , 9 SAME IN NEW BOOK

Page 118 Exercise 2B Questions 1 , 3 ← Do 9, 10, 11 of new book

New book p151 Ex9.1

Sum to *n* terms of an Arithmetic Sequence or Series

For the general arithmetic series with first term *a* and common difference *d*, the sum to *n* terms, partial sum, is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$ ← on formula sheet

Proof of above see p152/153 textbook

Illustration. For finite sequence: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

$$\text{Total}(10) = 1+2+3+4+\ldots+9+10$$
$$\text{Total}(10) = 10+9+8+7+\ldots+2+1$$ (adding)

$$2\times\text{Total}(10) = 11+11+11+11+\ldots+11+11$$

$$2\times\text{Total}(10) = 10\times 11$$

$$\text{Total}(10) = \frac{10}{2}\times 11$$

$$= \underline{55}$$

Remember $\text{Total}(n) = \frac{n}{2}(a+l)$, a = first term, l = last term

or $S_n = \frac{n}{2}(a+U_n)$, $U_n = a+(n-1)d$

$$= \frac{n}{2}(a + a + (n-1)d)$$

$$= \frac{n}{2}(2a+(n-1)d)$$

Examples

1) For the arithmetic series 3 + 7 + 11 + 15 + ……, find an expression for S_n, the sum to n terms and hence evaluate S_{20}

$S_n = \frac{n}{2}(a + l)$ $\quad$ $l = u_n = a + (n-1)d$

$= \frac{n}{2}(a + u_n)$

$S_n = \frac{n}{2}(2a + (n-1)d)$, $a = 3$, $d = 4$

$\Rightarrow S_n = \frac{n}{2}(6 + (n-1) \cdot 4)$

$= \frac{n}{2}(2 + 4n)$

$= n(1 + 2n)$

Hence $S_{20} = 20(1 + 2(20))$

$= 20(41)$

$= \underline{\underline{820}}$

2) Find the sum of the arithmetic series 17 + 19 + 21 +......+ 231

For number of terms

$U_n = a + (n-1)d$, $a = 17$ $d = 19-17$
$= 2$

$231 = 17 + (n-1) \cdot 2$

$231 = 15 + 2n$

$216 = 2n$

$n = 108$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{108} = 54(34 + 107 \times 2)$

$= 54 \times (34 + 214)$

$= 54 \times 248$

$= \underline{13392}$

3) In an arithmetic progression (general term) $u_{10} = 3$ and $S_6 = 76.5$. Find the smallest value of n such that $S_n < 0$.

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_6 = 3(2a + 5d) = 76.5$

$\Rightarrow 2a + 5d = 25.5 \quad (1)$

$U_n = a + (n-1)d$

$U_{10} = a + 9d = 3$

$\Rightarrow 2a + 18d = 6 \quad (2)$

Now $(2) - (1) \Rightarrow 13d = -19.5$

$\underline{d = -1.5}$

Given $a + 9d = 3$

$a = 3 - 9(-1.5)$

$= \underline{16.5}$

$S_n < 0$

$\frac{n}{2}(2a + (n-1)d) < 0$

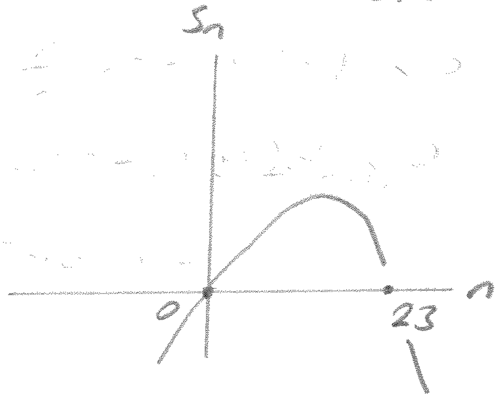
$\frac{n}{2}(33 - 1.5n + 1.5) < 0$

$n(34.5 - 1.5n) < 0$

$1.5n(23 - n) < 0$

$n < 0$ or $n > 23$ (from graph)

$S_n = 1.5n(23-n)$



For smallest n such that $S_n < 0 \Rightarrow \underline{\underline{n = 24}}$

Maths in Action Book 2 Page 120 Exercise 3A Questions1 , 2 , 3(a),(c),(e) ,4 , 6 , 8

Page 121 Exercise 3B Questions 1

Geometric Sequences

A geometric sequence or series is one in which there is a **common ratio**, *r*, between neighbouring terms i.e.

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} \quad \text{for all values of } n$$

for example 2 , 6 , 18 , 54,…… is a geometric sequence with **first term 2** and **common ratio 3**

$a, ar, ar^2, ar^3, ar^4, \ldots$

In general for a Geometric Sequence or Series, with first term *a* and common ratio *r*

$$u_n = ar^{n-1} \quad , r \neq 0, 1$$

$a_1 = 2$
$a_2 = 2 \times 3^1$
$a_3 = 2 \times 3^2$
$a_4 = 2 \times 3^3$
$a_n = 2 \times 3^{n-1}$

Examples

1. Show that 2sinx, sin2x and sin2xcosx could be the first three terms of a geometric sequence.

Let $a_1 = 2\sin x \quad a_2 = \sin 2x \quad a_3 = \sin 2x \cos x$

$$\frac{a_2}{a_1} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x$$

$$\frac{a_3}{a_2} = \frac{\sin 2x \cos x}{\sin 2x} = \cos x$$

OR
$r_1 = -$, $r_2 = -$
$r_1 = r_2 \Rightarrow$ common ratio etc.

Since $\frac{a_{n+1}}{a_n} = \cos x$ for $n = 1$ and $n = 2$

there is a common ratio for the first 3 terms.

2. Find the geometric sequence whose third term is 18 and whose eighth term is 4374.

$U_n = ar^{(n-1)}$

$U_3 = ar^2 = 18$

$U_8 = ar^7 = 4374$

Dividing to eliminate a gives

$r^5 = 243$

$r = 3$

$\Rightarrow a = 2$ Hence $\underline{U_n = 2(3)^{n-1}}$

3. Given the geometric sequence 3, 6, 12, 24, find n such that $u_n = 196608$

$a = 3, \quad r = \frac{6}{3} = 2$

Now $U_n = ar^{(n-1)}$

$196608 = 3(2)^{n-1}$

$65536 = 2^{(n-1)}$ $\quad y = a^x \Rightarrow \log_a y = x$

$\cancel{\log_2 65536 = \log_2 2^{(n-1)}}$

$\log_2 65536 = n-1$

$n = \log_2 65536 + 1$

$\underline{n = 17}$

Maths in Action Book 2 Page 123 Exercise 4A Questions 1(a),(d),(g), 2 , 3 , 6 , 7

New book p156 Ex9.3

Sum to *n* terms of a Geometric Sequence or Series

The sum of the terms in a geometric sequence is called a geometric series.

For the general geometric progression with first term a and common ratio r, the sum to n terms, (partial sum or finite sum) is given by

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1}$$

from

$$\begin{aligned}
S_n &= a + ar + ar^2 + \dots\dots\dots + ar^{n-1} \\
rS_n &= \quad ar + ar^2 + \dots\dots\dots + ar^{n-1} + ar^n \\
S_n - rS_n &= a - ar^n \\
(1-r)S_n &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

Examples

1) Find the 12th term and S_6 of the geometric sequence 1, 3, 9, 27, ...

$U_n = ar^{(n-1)}$, $a = 1$, $r = 3$

$\Rightarrow U_n = 3^{(n-1)}$

$\Rightarrow \underline{U_{12} = 3^{11} = 177147}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{1(1-3^6)}{1-3}$$

$$= \frac{-728}{-2}$$

$$= \underline{364}$$

2) What is the smallest number of terms that will give a total greater than 6000000 for the sequence 8, 24, 72,…..?

$S_n > 6000000$

$$\frac{a(1-r^n)}{1-r} > 6000000 \quad , \quad a=8 \; , \; r=3$$

$$\frac{8(1-3^n)}{-2} > 6000000$$

$$8(1-3^n) < -12000000$$

Switch inequality sign since $\times$ by negative

$$1-3^n < -1500000$$

$$-3^n < -1500001$$

$$3^n > 1500001$$

$$\log_3 3^n > \log_3 1500001$$

$$n > \log_3 1500001$$

$$n > 12.94\ldots$$

Hence $\underline{n=13}$ is the smallest no. of terms.

3) Find two possible geometric series such that $S_2 = 15$ and $S_4 = 255$

Geometric $\Rightarrow$ $a, ar, ar^2, ar^3, \ldots$

$S_2 = a + ar$
$= a(1+r)$

$S_4 = a + ar + ar^2 + ar^3$
$= a(1 + r + r^2 + r^3)$

Now $\dfrac{S_4}{S_2} = \dfrac{255}{15}$

$\Rightarrow$ Factorise Trinomial

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 1 & 1 \\ & & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

$f(-1) = 0$
$\Rightarrow (r+1)$ is a factor

$$\frac{\cancel{a}(1+r+r^2+r^3)}{\cancel{a}(1+r)} = 17$$

$$\frac{\cancel{(1+r)}(r^2+1)}{\cancel{(1+r)}} = 17$$

$r^2 + 1 = 17$

$r^2 = 16$

$r = \pm 4$

$r = -4$

$\Rightarrow a(-3) = 15$ from (1)

$a = -5$

$\Rightarrow U_n = ar^{n-1}$

$= -5(-4)^{n-1}$

$r = 4$

$\Rightarrow a(5) = 15$ from (1)

$a = 3$

$\Rightarrow U_n = ar^{n-1}$

$= 3(4)^{n-1}$

Maths in Action Book 2 Page 127 Exercise 5A Questions 1(a),(c),(e), 2(a),(c),(e)
3 , 4 , 6 , 7

New book p159 EX9.4

Sum To Infinity

If the partial sum S_n tends towards a limit as n tends to infinity, then the limit is called the sum to infinity of the series.

Arithmetic Series

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

can be written as

$$S_n = n^2\left[\frac{d}{2} + \frac{(a - \frac{d}{2})}{n}\right]$$

$$\text{and so as } n \to \infty \quad S_n \to n^2\frac{d}{2}$$
$$\text{and so as } n \to \infty \quad S_n \to \pm\infty \text{ depending on value of } d$$

The sum to infinity for an arithmetic series is undefined.

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r} \quad , r \neq 1$$

If $|r| > 1$ *then* $r^n \to \pm\infty$ *as* $n \to \infty$ Consider $3, 9, 27, 81, \ldots$ $r=3$

If $|r| < 1$ *then* $r^n \to 0$ *as* $n \to \infty$ Consider $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ $r=\frac{1}{3}$

If $r = 1$ *then* $S_n = na \to \pm\infty$ *as* $n \to \infty$ *depending on a* Consider $3, 3, 3, 3, 3, 3, \ldots$ $r=1$

If $r = -1$ *then* $S_n = 0$ *for even n and* $S_n = a$ *for odd a, as* $n \to \infty$ Consider $3, -3, 3, -3, 3, -3$

Hence

The **Sum to Infinity of a Geometric Series** is only defined when $|r| < 1$ and is given by

$$S_\infty = \frac{a}{1-r} \qquad -1 < r < 1$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a - ar^n}{1-r}$$

$$= \frac{a}{1-r} - \left(\frac{a}{1-r}\right) r^n$$

If $|r| < 1$ then as $n \to \infty$, $\left(\frac{a}{1-r}\right) r^n \to 0$

and $S_n \to \frac{a}{1-r}$

Examples

1) Express $0.0\dot{7}$ as a fraction

$0.0\dot{7} = 0.0777\ldots = 0.07 + 0.007 + 0.0007 + \ldots$

$r = \dfrac{0.007}{0.07} = 0.1$, $a = 0.07$

$S_\infty = \dfrac{a}{1-r}$, $|r| < 1$

$= \dfrac{0.07}{1-0.1}$

$= \dfrac{0.07}{0.9}$

$= \dfrac{7}{90}$

2) Find the sum to infinity of the geometric series 18 + 12 + 8 + …..

$a = 18 \qquad r = \frac{12}{18} = \frac{2}{3}$

$S_\infty = \frac{a}{1-r}$ since $|r| < 1$

$= \frac{18}{1-\frac{2}{3}}$

$= \frac{18}{\frac{1}{3}}$

$= \underline{\underline{54}}$

3) Given that 18 and 6 are two adjacent terms in an infinite geometric series with sum to infinity of 81, find the first term.

$r = \frac{6}{18} = \frac{1}{3}$

$S_\infty = \frac{a}{1-r}$, $|r| < 1$

$\Rightarrow \quad 81 = \frac{a}{1-\frac{1}{3}}$

$81 = \frac{a}{\frac{2}{3}}$

$81 \times \frac{2}{3} = a$

$a = 54$

Hence first term = $\underline{54}$

Maths in Action Book 2 Page 131 Exercise 6A Questions 1(a),(d), 2(a),(d)
3 , 4(a) , 6

Summary of Formula to Learn for NAB and Exam

Arithmetic Sequence

$$u_{n+1} - u_n = u_n - u_{n-1}$$

$$u_n = a + (n-1)d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Geometric Sequence

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}}$$

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$S_\infty = \frac{a}{1-r} \qquad for\ |r| < 1$$

Power Series

We can use polynomials to approximate some functios.

Any series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots$$

is called a **POWER SERIES.**

MacLaurins Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots\ldots$$

$f(x) = \sum_{i=0}^{\infty} \frac{f^{i}(0)}{i!}$

valid when $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$

Examples

1.(a) Find a power series for f(x) = cosx

(b) Hence find a power series for cos2x.

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

This then repeats since
$f^4(x) = \cos x$

Hence power series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$\cos x = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}\ldots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots$$

For cos 2x, replace x with 2x in f(x) = cosx

Hence

$$\cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \ldots$$

$$= 1 - \frac{2^2}{2}x^2 + \frac{2^4}{24}x^4 - \frac{2^6}{720}x^6 + \ldots$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots$$

Expansions for $\ln(1+x)$ and $\tan^{-1} x$

These can be found relatively easily from the simpler expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$$

Replace x with x^2 gives

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

Now $\ln(1+x) = \int (1+x)^{-1}\, dx$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + c$$

[Since when $x=0$, $\ln(1+x) = 0 \Rightarrow c = 0$]

Also $\tan^{-1} x = \int (1+x^2)^{-1}\, dx$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + c$$

[Again for $x=0$, $\tan^{-1} x = 0 \Rightarrow c = 0$]

2(a) Find the first four terms in MacLaurins series for f(x) $= e^{2x}$

(b) Hence find an expression for e^{2x+3}

(a) $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\Rightarrow e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots$

$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$

(b) $e^{2x+3} = e^{2x} \cdot e^3$ [Don't replace x with $2x+3$ as it does not produce a 'clean' power series]

$= e^3\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots\right)$

$= e^3 + 2e^3x + 2e^3x^2 + \frac{4e^3}{3}x^3 + \ldots$

3 Find the term in x^4 in the expansion of $e^x(1+2x)$.

$f(x) = (1+2x)e^x$

$= (1+2x)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$ only need to go this far for x^4

$= 1\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + 2x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$

term in $x^4 = 1\left(\frac{x^4}{4!}\right) + 2x\left(\frac{x^3}{3!}\right)$

$= \frac{1}{24}x^4 + \frac{2}{6}x^4$

$= \frac{3}{8}x^4$

valid for

Basic Expansions

sinx $= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$ $\quad x \in \mathbb{R}$

cosx $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$ $\quad x \in \mathbb{R}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \sum_{i=0}^{\infty} \frac{x^i}{i!}$ $\quad x \in \mathbb{R}$

ln(1 + x) $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \sum_{i=0}^{\infty} (-1)^i \cdot \frac{x^{i+1}}{i+1}$ $\quad -1 < x \leq 1$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad \sum_{i=0}^{\infty} (-1)^i \cdot \frac{x^{2i+1}}{2i+1}$ $\quad -1 < x < 1$

learn these

We can use these to derive others.

Examples

1. Find a power series for $\ln\frac{(1+x)}{(1-x)}$

using log laws

replacing x with $-x$

$$\ln\frac{(1+x)}{(1-x)} = \ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \ldots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \ldots \quad (-1 < x < 1)$$

2. Expand $e^{-2x}\sin 3x$ in ascending powers of x as far as x^4.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \sin x = x - \frac{x^3}{3!}$$

$$e^{-2x} = 1 - 2x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 \qquad \sin 3x = 3x - \frac{(3x)^3}{3!}$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 \qquad = 3x - \frac{9}{2}x^3$$

Now,

$$e^{-2x}\sin 3x = \left(3x - \frac{9}{2}x^3\right)\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4\right)$$

$$= 3x\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\right) - \frac{9}{2}x^3\left(1 - 2x + \ldots\right)$$

$$= 3x - 6x^2 + 6x^3 - 4x^4 - \frac{9}{2}x^3 + 9x^4 \ldots$$

$$= 3x - 6x^2 + 6x^3 - 4x^4 \quad (\text{as far as } x^4) \quad (x \in \mathbb{R})$$

3. Expand $e^{\sin x}$ as far as x^4.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \sin x = x - \frac{x^3}{3!}$$

Replacing x with sine x,

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{3!}\right) + \frac{\left(x - \frac{x^3}{3!}\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!}\right)^3}{3!} + \frac{\left(x - \frac{x^3}{3!}\right)^4}{4!}$$

$$= 1 + x - \frac{x^3}{6} + \frac{1}{2}\left(x^2 - \frac{2x^4}{3!} + \ldots\right) + \frac{1}{6}\left(x^3 + 3x^2\left(-\frac{x^3}{3!}\right) + \ldots\right) + \frac{1}{24}\left(x^4 + \ldots\right)$$

$$= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{6}x^4 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 \quad (\text{as far as } x^4) \quad (x \in \mathbb{R})$$