GAUSSIAN ELIMINATION

Simultaneous Equations

We already know how to solve two equations in two unknowns.

Now we are going to solve equations in three unknowns. To be able to do this we need three equations. We can then solve simultaneously by eliminating one letter at a time.

Example

Solve the system of equations $x + 2y - z = 7$ ①

$3x - y + 4z = -7$ ②

$2x + 3y + 2z = 4$ ③

Eliminate z (or x, or y)

① ×4 $4x + 8y - 4z = 28$
② ×1 $3x - y + 4z = -7$
$7x + 7y = 21$ ④

① ×2 $2x + 4y - 2z = 14$
③ ×1 $2x + 3y + 2z = 4$
$4x + 7y = 18$ ⑤

Now we have 2 equations with 2 unknowns

④ $7x + 7y = 21$
⑤ $4x + 7y = 18$
$3x = 3$
$x = 1$

From ④ $7 + 7y = 21$
$7y = 14$
$y = 2$

From ① $1 + 4 - z = 7$
$5 - z = 7$
$z = -2$

Solution

$x = 1,\ y = 2,\ z = -2$

Try,

$6x - 7y = 6$
$8x + 3y + 5z = 3$
$-5x - 3y + 2z = -7$

Solution
$x = 1,\ y = 0,\ z = -1$

To make this method more efficient we can put the equations into matrix form.

Example

Using the equations from the previous example

$$x + 2y - z = 7$$

$$3x - y + 4z = -7$$

$$2x + 3y + 2z = 4$$

In matrix form this gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 3 & -1 & 4 & -7 \\ 2 & 3 & 2 & 4 \end{array}\right]$$

This is called the **AUGMENTED MATRIX**

We now perform elementary row operations

- a row can be multiplied by a constant
- rows can be added or subtracted
- rows can be interchanged

We are aiming to get the matrix in the following form

$$\left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{array}\right]$$

x = number

get Zero's here

This is called **UPPER TRIANGULAR FORM**

We then revert back to equations and solve by back substitution.

This method of solving the equations is called **GAUSSIAN ELIMINATION**

So returning to the equations

$$x + 2y - z = 7$$

$$3x - y + 4z = -7$$

$$2x + 3y + 2z = 4$$

We solve as follows

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 3 & -1 & 4 & -7 \\ 2 & 3 & 2 & 4 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -7 & 7 & -28 \\ 0 & -1 & 4 & -10 \end{array}\right]$$

$R_2 \to -\frac{1}{7}R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & -1 & 4 & -10 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

show your intentions

use a 'leading' number to eliminate those below it.

changing back to equations...

R3: $3z = -6$
$z = -2$

R2: $y - z = 4$
$y + 2 = 4$
$y = 2$

R1: $x + 2y - z = 7$
$x + 4 + 2 = 7$
$x = 1$

Solution

$x = 1,\ y = 2,\ z = -2$

<u>Example</u>

Solve using Gaussian elimination

$$2y + 3z = 4$$
$$x + y + z = 2$$
$$4x + 2y + 3z = 6$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 3 & 6 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 4 & 2 & 3 & 6 \end{array}\right]$$

$R_3 \Rightarrow R_3 - 4R_1$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$R_3 \Rightarrow R_3 + R_2$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

change to equations

R_3: $2z = 2$
$z = 1$

R_2: $2y + 3z = 4$
$2y + 3 = 4$
$y = \frac{1}{2}$

R_1: $x + y + z = 2$
$x + \frac{1}{2} + 1 = 2$
$x = \frac{1}{2}$

<u>Solution</u>

$x = \frac{1}{2}, \; y = \frac{1}{2}, \; z = 1$

Tips for Success....

- State intentions at side of matrix
- Aim for $\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{array}\right)$
- then $\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{array}\right)$
- be very careful with negatives.

Maths in Action Book 1 Page 127 Exercise 4A Questions 1 , 2 ,3

Book solution wrong

Types of Solutions

All the examples we have looked at so far for solving 3 equations in 3 variables have **one unique solution** (i.e we get one answer for each of the letters)

This is not always the case and three possibilities occur :- **one unique solution**
inconsistent equations
redundant equation

(a) **One unique solution**

- occurs when we obtain one value for each variable, as seen so far
- Geometrical meaning – 3 planes meeting at a point. (see vectors unit 3)

(b) **Inconsistent equations**

- occur when a row in the matrix looks like

$$\begin{array}{ccc|c} 0 & 0 & 0 & k \end{array}$$

leading to e.g. :- $0 = 5$?

- Geometrical meaning – 3 planes not all meeting together eg 2 parallel planes, planes form prism (see vectors unit 3)

Inconsistent equations give NO SOLUTION

(c) **Redundant equation**

- occur when a row in the matrix looks like :-

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}$$

- Geometrical meaning – actually only two distinct equations meeting in a line.

A redundant equation leads to an INFINITE NUMBER OF SOLUTIONS

We have redundancy when one equation is 'built' from another eg. $3x + 2y = 7$ ① ② is built from ①
$6x + 4y = 14$ ②

Examples

Solve by Gaussian elimination

x +2y +2z = 11
2x – y + z = 8
3x + y + 3z =18

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right]$$

$$\begin{array}{l} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - 3R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & -5 & -3 & -15 \end{array}\right]$$

$$R_3 \Rightarrow R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$0 = -1$ is not possible

Hence the system is inconsistent

$\Rightarrow$ NO SOLUTIONS.

<u>Example</u>
Solve by Gaussian Elimination

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{array}\right]$$

$R_2 \Rightarrow R_2 - R_1$
$R_3 \Rightarrow R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & 3 & -9 \end{array}\right]$$

$R_3 \Rightarrow R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution

$$x = \frac{27 - 8z}{3}$$

$$y = \frac{z + 3}{3}$$

$$z = z$$

There is a <u>redundant</u> row $\Rightarrow$ infinite solutions

use equations to find a general solution in terms of z

Given any z,

from R_2: $-3y + z = -3$

$$3y = z + 3$$

$$y = \frac{z+3}{3}$$

From R_1: $x + 2y + 2z = 11$

$$x = 11 - 2\left(\frac{z+3}{3}\right) - 2z$$

$$3x = 33 - 6 - 2z - 6z$$

$$x = \frac{27 - 8z}{3}$$

Exam Question Example

Use Gaussian elimination to reduce the system of equations

$$\begin{aligned} 2x - y + \alpha z &= 1 \\ x - y + 2z &= -3 \\ -x + 2y - 3z &= 2 \end{aligned}$$

to upper triangular form.

Explain what happens when $\alpha = 3$.

Find the solution corresponding to $\alpha = -13$.

$$\left[\begin{array}{ccc|c} 2 & -1 & \alpha & 1 \\ 1 & -1 & 2 & -3 \\ -1 & 2 & -3 & 2 \end{array}\right]$$

$R_1 \Leftrightarrow R_3$ $$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 2 \\ 1 & -1 & 2 & -3 \\ 2 & -1 & \alpha & 1 \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 + 2R_1$ $$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & \alpha-6 & 5 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$ $$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \alpha-3 & 8 \end{array}\right]$$

Solution $x = -\frac{7}{2}$
$y = -\frac{3}{2}$
$z = -\frac{1}{2}$

When $\alpha = 3$

$R_3 \Rightarrow (\alpha - 3)z = 8$
$\Rightarrow 0 = 8$

$\Rightarrow$ equations are inconsistent.
$\Rightarrow$ NO SOLUTIONS

When $\alpha = -13$

R_3: $-16z = 8$
$z = -\frac{1}{2}$

R_2: $y - z = -1$
$y + \frac{1}{2} = -1$
$y = -\frac{3}{2}$

R_1: $-x + 2y - 3z = 2$
$-x - 3 + \frac{3}{2} = 2$
$-x - \frac{3}{2} = \frac{4}{2}$
$x = -\frac{7}{2}$

Tips for success .....

- For equations containing an unknown coefficient eg α, solve first in terms of the letter. eg find x, y, z in terms of α.
- redundant row $0\,0\,0\,|\,0 \Rightarrow$ infinite solutions (write x y in terms of z)
- inconsistent row $0\,0\,0\,|\,k \Rightarrow$ no solutions.

Exercise :- MiA AH1, page 130, Exercise 5. Questions 1(a) , (c) , (f) , 2 , 3 , 4

The Stability of Gaussian Elimination

For a method to be stable, if a small error is introduced (eg rounding) there must only be a small change in the solution.

Consider $100x + 99y = 199$ ①
$99x + 98y = 197$ ②

① × 99 $9900x + 9801y = 19701$
② × 100 $9900x + 9800y = 19700$

$y = 1$

From ① $100x + 99 = 199$
$\Rightarrow x = 1$

Now changing the original equations the original equations slightly

③ $100x + 99y = 200$ ← 199 ± 1 $\left(\%\text{ error} = \frac{1}{199} \times 100\% = 0.5\%\right)$
④ $99x + 98y = 197$

③ × 99 $9900x + 9801y = 19800$
④ × 100 $9900x + 9800y = 19700$

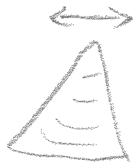
$y = 100$

From ③ $100x + 9900 = 200$
$100x = -9700$
$x = -97$

Minimum error in x is 1 ± 98 $\left(\%\text{ Error} = \frac{98}{1} \times 100\% = 9800\%\right)$

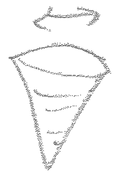
A small change has drastically changed the solution. We say these equations are **ill-conditioned.**

A stable problem is well conditioned.



Stable cone
unaffected by small changes

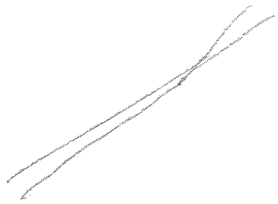
An unstable problem is ill conditioned.



unstable cone

Note Ill conditioning in our example is due to the lines being near parallel

$$m_1 = \frac{100}{99} = 1.\dot{0}\dot{1}0101\ldots \qquad m_2 = \frac{99}{98} = 1.010204\ldots$$



2012 Exam Paper Question 14