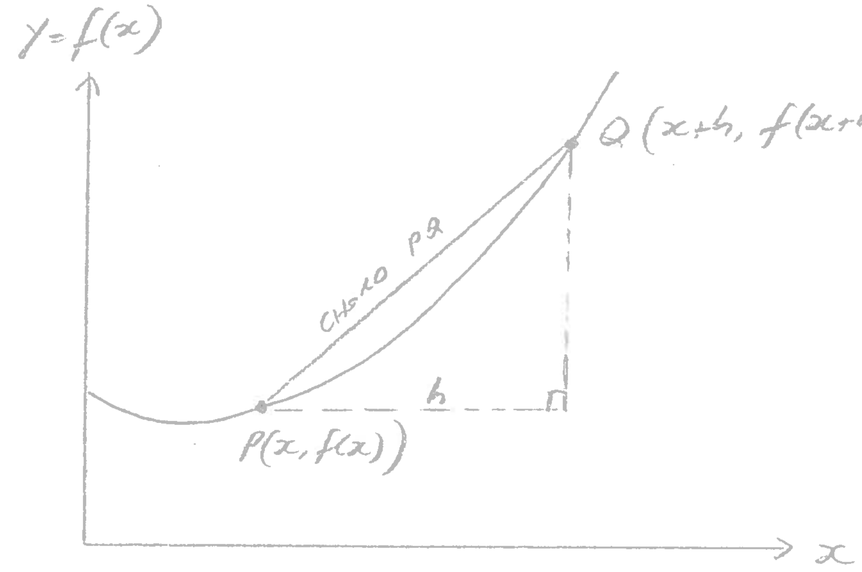
DIFFERENTIATION

Differentiation from First Principles



From the graph we see that the rate of change of *f(x)* at point P can be estimated using chord PQ. As Q approaches P, $h \to 0$, and the gradient of PQ reaches **a limit** which is equal to the gradient of the tangent to the curve of $y = f(x)$.
This limit is denoted by $f'(x)$ and is called the derivative (or the rate of change) of *f* at *x*.

We can see :-

$$m_{PQ} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

Now

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

As Q approaches P from the right, and the limit f ' (x) exists, we find the right derivative. If Q approaches P from the left and the limit f ' (x) exists then we find the left derivative. If both limits exist and are the same then we can conclude that the function is differentiable.

Example 1

Differentiate $f(x) = x^2 + 1$, from first principles

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(x+h)^2 + 1 - (x^2+1)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \right)$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= \underline{\underline{2x}}$$

Rules for Limits

1) $If\ f(x) = c\,,\quad then\ \lim_{x\to a} f(x) = c$

2) $\lim_{x\to a} kf(x) = k \times \lim_{x\to a} f(x)$

3) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

4) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$

5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\qquad provided\ \lim_{x\to a} g(x) \neq 0$

The following rules are needed to differentiate $\sin x$ and $\cos x$ .....

6) $\lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$

7) $\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) = 0$

Example 2

Differentiate $f(x) = \sin x$, from first principles

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(x+h) - \sin x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= \sin x \times 0 + \cos x \times 1$$

$$= \underline{\underline{\cos x}}$$

Exercise :- MiA AH1 – page 29, Ex.1A (not Q2,4); Ex1B (not Q6,8)

Recall Notation

$f(x) = x^2 + \sin x$ $\qquad$ $f'(x) = 2x + \cos x$

$y = x^2 + \sin x$ $\qquad$ $\dfrac{dy}{dx} = 2x + \cos x$

$$\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x$$

Rules for Differentiation

Differentiation of Sums and Differences

From study in Higher we saw that writing functions as sums and differences makes differentiation simple.

e.g. $\quad y = \dfrac{(x+3)(x-2)}{x^2}$

$$= \frac{x^2 + x - 6}{x^2}$$

$$= \frac{x^2}{x^2} + \frac{x}{x^2} - \frac{6}{x^2}$$

$$= 1 + x^{-1} - 6x^{-2} \leftarrow \text{Easy to differentiate.}$$

Standard Derivatives

From Higher we know that :-

$$\frac{d(ax^n)}{dx} = nax^{n-1}$$ ← Try proving this from first principles (Need binomial theory first)

$$\frac{d(\sin(ax+b))}{dx} = a\cos(ax+b)$$

$$\frac{d(\cos(ax+b))}{dx} = -a\sin(ax+b)$$

MIA AH 1 Pg 30 Ex 2 or Revision sheet.

The Chain Rule

The Chain Rule is used when differentiating composite functions – i.e. functions in the form $y = f(g(x))$ or $y = u(v(x))$. It is sometimes known as the "function of a function" rule.

Some functions may be differentiated by inspection:-

e.g. $y = (x^2 - 3x)^4$

Remember: "Differentiate the bracket, then multiply by the derivative of the contents of the bracket" - from Higher ?

$$\frac{dy}{dx} = 4(x^2-3x)^3 \cdot \frac{d}{dx}(x^2-3x)$$

$$= \underline{4(x^2-3x)^3 \cdot (2x-3)}$$

Other functions require a more formal substitution:

e.g. If $y = \dfrac{2}{\sqrt{(x^2-3x+1)^3}}$, find $\dfrac{dy}{dx}$

$Y = 2u^{-\frac{3}{2}}$ where $u = (x^2 - 3x + 1)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -3u^{-\frac{5}{2}} \cdot (2x-3)$$

$$= -\frac{3(2x-3)}{(x^2-3x+1)^{\frac{5}{2}}}$$

The Chain Rule may be thought of as :-

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is easily adapted to more complicated functions:-

Examples Find the derivatives of :-

1. $y = \sin^3(2x - \frac{\pi}{3})$

2. $y = \frac{1}{\cos^2 4x}$ $= (\cos 4x)^{-2}$

Solutions

1) $y = u^3$, $u = \sin v$, $v = 2x - \frac{\pi}{3}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= 3u^2 \cdot \cos v \cdot 2$

$= 6 \sin^2 v \cos v$

$= 6 \sin^2 (2x - \frac{\pi}{3}) \cos (2x - \frac{\pi}{3})$

---

2) $y = u^{-2}$ $u = \cos v$ $v = 4x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= -2u^{-3} \cdot (-\sin v) \cdot 4$

$= -8 (\cos v)^{-3} (-\sin v)$

$= \frac{8 \sin 4x}{\cos^3 4x}$

Exercise :- MiA AH1 – pages 32-33, Exercise 3A (not Q4) pages 33-34, Exercise 3B

The Product Rule

Consider the function, $y = x^2 \sin x$. How would we find its derivative ?

Clearly this function is a **product** and is of the form $y = uv$, where u and v are both functions of x. (i.e. $y = u(x).v(x)$)

$y = uv$ . . . .(1)   If u and v are functions of x, and y changes by a small amount δx (or h)

$y + \delta y = (u + \delta u)(v + \delta v)$ . . . .(2)

(2) – (1) $\delta y = (u + \delta u)(v + \delta v) - uv$

$= uv + u\delta v + v\delta u + \delta u \delta v - uv$

$= u\delta v + v\delta u + \delta u \delta v$

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

As $h \to 0$ ($\therefore \delta x \to 0$)

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ And if $\delta x \to 0$,

$\frac{\delta u}{\delta x} \to \frac{du}{dx}$ δu and $\delta v \to 0$

$\frac{\delta v}{\delta x} \to \frac{dv}{dx}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This is known as the **PRODUCT RULE**

In alternate notations:

h(x) = f(x).g(x) $\Rightarrow$ h′(x) = f′(x).g(x) + f(x).g′(x)

y = uv $\Rightarrow$ y′ = u′v + uv′   where $y' = \frac{dy}{dx}$, $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$

Examples: Find derivatives of :-

1. $y = x^2 \sin x$

2. $y = (3x+4)^4 \cos(2x - \frac{\pi}{4})$

Solutions:

1) $u = x^2$, $v = \sin x$

$u' = 2x$, $v' = \cos x$

$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$

$= x(2\sin x + x\cos x)$

Formal substitution may not be needed in 'simple' cases.

2) $u = (3x+4)^4$, $v = \cos(2x - \frac{\pi}{4})$

$u' = 12(3x+4)^3$, $v' = -2\sin(2x - \frac{\pi}{4})$

$\frac{dy}{dx} = u'v + uv'$

$= 12(3x+4)^3 \cos(2x - \frac{\pi}{4})$

$-2(3x+4)^4 \sin(2x - \frac{\pi}{4})$

$= 2(3x+4)^3 \left[6\cos(2x - \frac{\pi}{4}) - (3x+4)\sin(2x - \frac{\pi}{4})\right]$

Take out a common factor to aid later simplification.

Exercise :- MiA AH1 – pages 35-36, Exercise 4A
- page 36, Exercise 4B

The Product Rule Extended

Find the derivative where y = uvw

y = (uv)w

y' = (uv)'w + (uv)w'

= (u'v + uv')w + uvw'

= u'vw + uv'w + uvw'

Example

Differentiate $y = x^2(x+1)^2\sin x$

$y = u \cdot v \cdot w$

$u = x^2 \quad v = (x+1)^2 \quad w = \sin x$

$u' = 2x \quad v' = 2(x+1) \quad w' = \cos x$

$\frac{dy}{dx} = u'vw + uv'w + uvw'$

$= 2x(x+1)^2\sin x + x^2 \cdot 2(x+1)\sin x + x^2(x+1)^2\cos x$

$= x(x+1)\left[2(x+1)\sin x + 2x\sin x + x(x+1)\cos x\right]$

$= x(x+1)\left[2x\sin x + 2\sin x + 2x\sin x + x^2\cos x + x\cos x\right]$

$= x(x+1)\left[2\sin x(2x+1) + x\cos x(x+1)\right]$

$= x(x+1)\left[2(2x+1)\sin x + x(x+1)\cos x\right]$

Again, no formal substitution is necessary in 'simple' cases.

The Quotient Rule

How can we differentiate $y = \tan x$?

$y = \tan x$ can be rewritten as $y = \dfrac{\sin x}{\cos x}$ which is a **quotient** in the form $y = \dfrac{u}{v}$ where u and v are both functions of x.

$$y = \frac{u}{v}$$

$y = uv^{-1}$ which can be differentiated using the Product Rule

$$\frac{dy}{dx} = u\frac{dv^{-1}}{dx} + v^{-1}\frac{du}{dx}$$

$\dfrac{d(v^{-1})}{dx} = \dfrac{d(v^{-1})}{dv}\dfrac{dv}{dx}$ by the Chain Rule

$$\frac{dy}{dx} = u \times -v^{-2}\frac{dv}{dx} + v^{-1}\frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{u}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{u}{v^2}\frac{dv}{dx} + \frac{v}{v^2}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This is known as the **QUOTIENT RULE**

Alternate notations:

$h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x).g(x) - f(x).g'(x)}{[g(x)]^2}$

$y = \dfrac{u}{v} \quad \Rightarrow \quad y' = \dfrac{u'v - uv'}{v^2}$

Examples: Find derivatives of the following :-

1. $y = \dfrac{\sin x}{x^2}$ 2. $y = \dfrac{x^2}{\sin x}$ 3. $y = \tan x$ 4. $y = \dfrac{(x+1)^2(x-3)}{x+3}$

Solutions:

1) $u = \sin x \quad v = x^2$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{\cos x \cdot x^2 - \sin x \cdot 2x}{(x^2)^2}$$

$$= \frac{x(x\cos x - 2\sin x)}{x^4}$$

$$= \frac{x\cos x - 2\sin x}{x^3}$$

2) $u = x^2 \quad v = \sin x$

$$\frac{dy}{dx} = \frac{2x\sin x - x^2\cos x}{\sin^2 x}$$

3) $y = \tan x$

$$= \frac{\sin x}{\cos x}$$

$u = \sin x \quad v = \cos x$

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \underline{\underline{\frac{1}{\cos^2 x}}}$$

4) $u = (x+1)^2(x-3)$ $\qquad v = x+3$

$u' = 2(x+1)(x-3) + (x+1)^2 \cdot 1$ $\qquad v' = 1$

$= (x+1)\left[2(x-3) + (x+1)\right]$

$= (x+1)(3x-5)$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{(x+1)(3x-5)(x+3) - (x+1)^2(x-3)}{(x+3)^2}$$

$$= \frac{(x+1)\left[(3x-5)(x+3) - (x+1)(x-3)\right]}{(x+3)^2}$$

$$= \frac{(x+1)\left[3x^2+4x-15 - (x^2-2x-3)\right]}{(x+3)^2}$$

$$= \frac{(x+1)(2x^2+6x-12)}{(x+3)^2}$$

$$= \frac{2(x+1)(x^2+3x-4)}{(x+3)^2} = \frac{2(x+1)(x+4)(x-1)}{(x+3)^2}$$

Exercise :- MiA AH 1, pages 37-38, Exercise 5A and Exercise 5B

More Standard Derivatives

By definition we "rename" $\frac{1}{\cos x}$, $\frac{1}{\sin x}$ and $\frac{1}{\tan x}$ for convenience such that :-

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Note

$$\tan^2 x + 1 = \sec^2 x \qquad\qquad 1 + \cot^2 x = \operatorname{cosec}^2 x$$

Proof

$LHS = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$LHS = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}$

$= \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x$

Use the chain rule (or the quotient rule) to find the derivatives of sec x, cosec x and cot x.
Find also the derivative of tan x.

If y = sec x $\frac{dy}{dx}$ = sec x tan x	If y = cosec x $\frac{dy}{dx}$ = -cot x cosec x
If y = cot x $\frac{dy}{dx}$ = - cosec^2 x	If y = tan x $\frac{dy}{dx}$ = $\sec^2$ x

Examples:-

Find derivatives of:

(a) y = 2 sec (2x - $\frac{\pi}{4}$)

$y = 2\sec u \quad , \quad u = 2x - \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 2\sec u \tan u \cdot 2$

$= 4\sec\left(2x - \frac{\pi}{4}\right)\tan\left(2x - \frac{\pi}{4}\right)$

(b) y = $\tan^2$ 3x

$y = (\tan 3x)^2$

$= 2\tan 3x \cdot \sec^2 3x \cdot 3$

$= 6\tan 3x \sec^2 3x$

Exercise:- MiA AH1 page 40 Ex 7 , NOT Q1 or 4(b)

More Standard Derivatives

The Exponential Function, exp(x) or e^x

Define $y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots$ see Sequences and Series later.

Try for a few different values of x.

Differentiating:-

$$\frac{dy}{dx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots.$$ which is identical to y.

Hence, if

$$\boxed{\begin{aligned} &y = e^x \\ &\Rightarrow \frac{dy}{dx} = e^x \end{aligned}}$$

The Logarithmic Function, $y = \ln x$

Given $y = \ln x \quad \Leftrightarrow \quad x = e^y$

differentiating w.r.t. y: $\frac{dx}{dy} = e^y$

but $\frac{dy}{dx} = \frac{1}{dx/dy}$

$\therefore \frac{dy}{dx} = \frac{1}{e^y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$

Hence, if

$$\boxed{\begin{aligned} &y = \ln x \\ &\Rightarrow \frac{dy}{dx} = \frac{1}{x} \end{aligned}}$$

Examples:-

Find derivatives of:

(a) $y = \exp(5x^2)$ (b) $y = e^{2x}\sin 2x$ (c) $y = \ln(5x^2 - 6)$ (d) $y = \ln(\frac{x+2}{x+3})$

(a) $y = e^{(5x^2)}$

$\frac{dy}{dx} = e^{(5x^2)} \cdot \frac{d}{dx}(5x^2)$

$= \underline{\underline{10x\, e^{(5x^2)}}}$

(b) $u = e^{2x}$ $v = \sin(2x)$

$u' = 2e^{2x}$ $v' = 2\cos 2x$

$\frac{dy}{dx} = 2e^{2x}\sin(2x) + 2e^{2x}\cos 2x$

$= \underline{\underline{2e^{2x}(\sin 2x + \cos 2x)}}$

(c) $y = \ln(5x^2 - 6)$

$\frac{dy}{dx} = \frac{1}{5x^2-6} \cdot \frac{d}{dx}(5x^2-6)$

$= \underline{\underline{\frac{10x}{5x^2-6}}}$

(d) $y = \ln\left(\frac{x+2}{x+3}\right)$ use log rules

$= \ln(x+2) - \ln(x+3)$

$\frac{dy}{dx} = \frac{1}{x+2} - \frac{1}{x+3}$

$= \frac{(x+3)-(x+2)}{(x+2)(x+3)}$

$= \frac{1}{(x+2)(x+3)}$

Exercise:- MiA AH1, pages 43-44, Exercise 8A, Q1, 2, 3, 4a,c,e, 5a,b,e, 6a,c,d
Exercise 8B, Q1,2

Inverse Functions

The derivative of an inverse function can be found by the technique shown below.

* If $f(x) = e^x$ then the inverse function $f^{-1}(x) = \ln x$.
We can find the derivative of $f^{-1}(x)$ as follows.

$y = \ln x$

$\Rightarrow x = e^y$

$\frac{dx}{dy} = e^y$

So $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

? Or use

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

maybe best not
only inverse trig functions in syllabus

This technique can be applied to find the derivatives of the inverse trig functions.

$y = \sin^{-1} x \qquad y = \cos^{-1} x \qquad y = \tan^{-1} x$

Proof
Find the derivative for $y = \sin^{-1} x$

$y = \sin^{-1} x$

$\Rightarrow x = \sin y$

$\frac{dx}{dy} = \cos y$

So $\frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{1}{\cos(\sin^{-1} x)}$

$= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}}$

$= \frac{1}{\sqrt{1 - x^2}}$

Identity

$\sin^2 A + \cos^2 A = 1$

$\Rightarrow \cos A = \sqrt{1 - \sin^2 A}$

Note: $\sin(\sin^{-1} x) = x$

Three standard derivatives

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Examples

Find the derivative of

1. $y = \sin^{-1}(2x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \cdot \frac{d}{dx}(2x)$$
$$= \frac{2}{\sqrt{1-4x^2}}$$

2. $y = \tan^{-1}(e^{2x})$

$$\frac{dy}{dx} = \frac{1}{1+(e^{2x})^2} \cdot \frac{d}{dx}(e^{2x})$$
$$= \frac{2e^{2x}}{1+e^{4x}}$$

3. $y = \ln(\cos^{-1} e^x)$

$$u = \cos^{-1}(e^x)$$
$$\frac{du}{dx} = -\frac{1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x)$$
$$= -\frac{e^x}{\sqrt{1-e^{2x}}}$$

$$= \ln u$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$
$$= -\frac{e^x}{\sqrt{1-e^{2x}} \cdot \cos^{-1}(e^x)}$$

MiA AH2 Page 32 Exercise 2 Every second question.

4. $y = x^2 \tan^{-1} x$

$u = x^2$ $\quad v = \tan^{-1} x$

$u' = 2x$ $\quad v' = \dfrac{1}{1+x^2}$

$$\frac{dy}{dx} = 2x \tan^{-1} x + \frac{x^2}{1+x^2}$$ [Product rule]

5. $y = \dfrac{e^{2x}}{\sin^{-1} x}$

$u = e^{2x}$ $\quad v = \sin^{-1} x$

$u' = 2e^{2x}$ $\quad v' = \dfrac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{2e^{2x} \sin^{-1} x - \dfrac{e^{2x}}{\sqrt{1-x^2}}}{(\sin^{-1} x)^2}$$

$$= \frac{2e^{2x}\sqrt{1-x^2}\,\sin^{-1} x - e^{2x}}{\sqrt{1-x^2}\,(\sin^{-1} x)^2}$$

MiA AH2 Pages 33/34 Exercise 3A questions 1, 2a,b,d, 3a,d, 4a,b, 5a, 6
Exercise 3B questions 1, 5b,c,d,f

Implicit Functions

The three formulae shown below are all different versions of the same linear equation:-

$y = \frac{2}{2}x - 5$ Here, the function, y, is written **explicitly** in terms of x.

$x = \frac{3}{2}y + \frac{15}{2}$ Here the function, x, is written **explicitly** in terms of y.

$2x - 3y - 15 = 0$ This is not explicit in any variable, but y is still a function in x, and vice versa. This function is written **implicitly.**

Many functions are written implicitly (e.g. the equations of circles), and hence it is useful to be able to differentiate implicit functions, **with respect to a single variable.**

N.B. You will use the chain and product rules often in implicit differentiation.

Examples

1. Find $\frac{dy}{dx}$ in each of the following :-

(a) $x^2 + y^2 = 64$ (b) $3x^2 + 7xy + 9\sin y = 6$

USE PRODUCT RULE

(a) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(64)$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$2y \cdot \frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$

(b) $\frac{d}{dx}(3x^2 + 7xy + 9\sin y) = \frac{d}{dx}(6)$

$6x + 7y + 7x \cdot \frac{dy}{dx} + 9\cos y \cdot \frac{dy}{dx} = 0$

$(7x + 9\cos y)\frac{dy}{dx} = -6x - 7y$

$\frac{dy}{dx} = \frac{-(6x + 7y)}{7x + 9\cos y}$

2. Find the equation of the tangent to the curve with equation $2x^2 - 3xy - y^2 = 1$ at the point with co-ordinates (2,1)

$$2x^2 - 3xy - y^2 = 1$$

$$4x - 3y - 3x\frac{dy}{dx} - 2y\cdot\frac{dy}{dx} = 0$$

$$4x - 3y = (3x + 2y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x - 3y}{3x + 2y}$$

At $\overset{x\ y}{(2,1)}$ $\quad m_{tan} = \dfrac{4(2) - 3(1)}{3(2) + 2(1)}$

$$= \frac{5}{8}$$

For equation: $y - b = m(x - a)$

$$y - 1 = \frac{5}{8}(x - 2)$$

$$8y - 8 = 5x - 10$$

$$8y = 5x - 2$$

$$\underline{\underline{y = \frac{5}{8}x - \frac{1}{4}}}$$

Exercise :- **MiA AH2,** **page 36; Exercise 4A, questions 1, 3, 4, 5, 8.**
page 37; Exercise 4B, questions 1, 2, 4.

* Facing page to be done first!!

Logarithmic Differentiation

Logarithmic differentiation is used to differentiate functions when x appears as a power.

It can also be used for complicated functions containing powers, roots products and quotients.

Method

- Take logs of both sides
- Simplify using rules of logs
- Differentiate both sides
- Rearrange to get $\frac{dy}{dx}$

Examples

1. Differentiate $y = 4^x$

$$\ln y = \ln 4^x$$
$$\ln y = x \cdot \ln 4$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 4$$
$$\frac{dy}{dx} = y \ln 4$$
$$= \underline{\underline{4^x \ln 4}}$$

2. Differentiate $y = x^x$

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} \qquad [\text{Product Rule}]$$
$$\frac{dy}{dx} = \underline{\underline{x^x(\ln x + 1)}}$$

Second Derivatives of Implicit Functions

The second derivative may be used to give us information about the stationary points of functions. This also applies to functions given implicitly.

Example

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the implicit function $x^2 + 7xy + 9y^2 = 6$

$$x^2 + 7xy + 9y^2 = 6$$

$$2x + 7y + 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = 0 \quad *$$

$$(7x + 18y)\frac{dy}{dx} = -2x - 7y$$

$$\frac{dy}{dx} = \frac{-2x - 7y}{7x + 18y}$$

Note: $\frac{d^2y}{dx^2}$ **may be found by differentiating** $\frac{dy}{dx}$.

However, it is often more convenient to differentiate each term at this (*) stage.
It is easier to find derivatives of sums and differences than say, use the quotient rule.

From * $\quad 2x + 7y + 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = 0$

Now once again finding $\frac{d}{dx}$ (both sides)

$$2 + 7\frac{dy}{dx} + \left[7\frac{dy}{dx} + 7x\frac{d^2y}{dx^2}\right] + 18\left(\frac{dy}{dx}\right)^2 + 18y\frac{d^2y}{dx^2} = 0$$

$$(7x+18y)\frac{d^2y}{dx^2} = -2 - 14\frac{dy}{dx} - 18\left(\frac{dy}{dx}\right)^2$$

$$(7x+18y)\frac{d^2y}{dx^2} = -2\left(1 + 7\frac{dy}{dx} + 9\left(\frac{dy}{dx}\right)^2\right)$$

$$(7x+18y)\frac{d^2y}{dx^2} = -\frac{2}{7x+18y}\left(1 + 7\frac{dy}{dx} + 9\left(\frac{dy}{dx}\right)^2\right)$$

$$\frac{d^2y}{dx^2} = -\frac{2}{7x+18y}\left(1 + 7\left(\frac{2x+7y}{7x+18y}\right) + 9\left(\frac{2x+7y}{7x+18y}\right)^2\right)$$

Note that $\frac{dy}{dx}$ is itself a function of x, so the product rule is needed at the first step.

The result here could be further simplified however in most cases we just need to be able to substitute in values for x and y and need not become bogged down in algebraic simplification.

Exercise MiA AH Book2 page 38 Exercise 5 Questions 1a, b, c, e, h, i, 4, 6, 7, 8

3. Differentiate $y = \dfrac{x^2\sqrt{7x-3}}{1+x}$

$$\ln y = \ln\left(\frac{x^2\sqrt{7x-3}}{1+x}\right)$$

$$\ln y = \ln x^2 + \ln(7x-3)^{\frac{1}{2}} - \ln(1+x)$$

$$\ln y = 2\ln x + \tfrac{1}{2}\ln(7x-3) - \ln(1+x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2}\cdot\frac{7}{7x-3} - \frac{1}{1+x}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{7}{2(7x-3)} - \frac{1}{(1+x)}\right)$$

$$= y\left(\frac{2\times2(7x-3)(1+x) + 7x(1+x) - 2x(7x-3)}{2x(7x-3)(1+x)}\right)$$

$$= y\left(\frac{4(7x^2+4x-3) + 7x + 7x^2 - 14x^2 + 6x}{2x(7x-3)(1+x)}\right)$$

$$= \frac{x^2\sqrt{7x-3}}{1+x}\cdot\frac{21x^2+29x-12}{2x(7x-3)(1+x)}$$

$$= \frac{x(21x^2+29x-12)}{2\sqrt{7x-3}\,(1+x)^2}$$

Exercixe MiA AH Book2 page 40 Exercise 6 Questions 1, 2, 4, 5, 7, 8, 10b

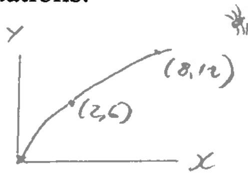
Parametric Equations

The position of an object moving on a plane is often given in terms of the object's x- and y- coordinates.

Obviously the position of the object may vary with time; hence x and y will be functions of time, and the object's position mat be defined by 2 equations.

ant crawling

Suppose:-



$x = 2t^2$ and $y = 6t$

The position of the object may be calculated by substitution for various values of t as shown:-

$P(t)$: $x = 2t^2$, $y = 6t$
$P(0)$: $x = 0$, $y = 0$ $\Rightarrow P(0,0)$
$P(1)$: $x = 2$, $y = 6$ $\Rightarrow P(2,6)$
$P(2)$: $x = 8$, $y = 12$ $\Rightarrow P(8,12)$, etc.

The original equations for x and y are known as the **parametric equations** of the object, with t being the parameter.

? If we wish to determine the velocity or acceleration of the object it can be useful to eliminate the parameter to gain the **constraint equation.**

Example

From above $x = 2t^2$ $\qquad y = 6t$

$$\Rightarrow t = \frac{y}{6} \quad \text{..... sub into the equation for } x$$

$x = 2t^2$ $\qquad \Rightarrow x = 2\times\left(\frac{y}{6}\right)^2$

$$x = \frac{y^2}{18}$$

rearranging $\qquad \Rightarrow y^2 = 18x$

Exercise **MiA AH2, page 42, Exercise 7A Q1.**

Parametric Differentiation

It is not always necessary to obtain the constraint equation to in order to obtain derivatives from parametric equations. **In fact it is an advantage to do otherwise and directly determine the derivative in terms of the parameter.**

Suppose a function is defined as the parametric equations:-

$$x = x(t) \quad \text{and} \quad y = y(t)$$

then $\quad \dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} \qquad$ and remembering $\dfrac{dx}{dt} = \dfrac{1}{dt/dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Examples

1. Find $\dfrac{dy}{dx}$ on the curve defined by :- $x = 3t - 2$ and $y = 4t^2$

$\dfrac{dx}{dt} = 3 \qquad \dfrac{dy}{dt} = 8t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

$= \dfrac{8t}{3}$

2. Find $\frac{dy}{dx}$ on the curve defined by :- $x = e^{\theta} \sin\theta$ and $y = 4\cos 2\theta$

$$\frac{dx}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta$$

$$\frac{dy}{d\theta} = -4\sin 2\theta \cdot \frac{d}{dx}(2\theta)$$
$$= -8\sin 2\theta$$

So $$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{-8\sin 2\theta}{e^{\theta}(\sin\theta + \cos\theta)}$$

Exercise **Sheet, Exercise 5, ALL questions**

Parametric Differentiation – Second Derivatives

The second derivative of any function is found by differentiating the derivative.

So for a parametric function defined by :-

$$x = x(t) \quad \text{and} \quad y = y(t)$$

$\dfrac{dy}{dx}$ is a function of t, or $\dfrac{dy}{dx} = f(t)$

Hence $\dfrac{d^2y}{dx^2}$ is the derivative of $f(t)$, or $f'(t)$ sticking with this notation

More conventionally:-

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \text{ provided } \frac{dy}{dx} \text{ is in terms of } x.$$

However, in our case $\dfrac{dy}{dx}$ is in terms of the parameter t.

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} && \text{using the Chain Rule} \\ &= \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}\end{aligned}$$

$$\boxed{\therefore \frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}} \quad \text{or} \quad \boxed{\therefore \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}}$$

Examples

1. For the curve defined by:-

$$x = a\cos\theta \quad \text{and} \quad y = a\sin\theta\,, \quad \text{find} \quad \frac{dy}{dx} \quad \text{and} \quad \frac{d^{2y}}{dx^2}$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{a\cos\theta}{-a\sin\theta}$$

$$= -\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d(-\cot\theta)}{d\theta} \cdot \frac{d\theta}{dx}$$

Using chain rule since θ is a function of x

ie. $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$

$$= -\left(-\operatorname{cosec}^2\theta\right) \cdot \frac{1}{-a\sin\theta}$$

$$= \operatorname{cosec}^2\theta \cdot \frac{1}{-a} \cdot \operatorname{cosec}\theta$$

$$= \underline{\underline{-\frac{1}{a}\operatorname{cosec}^3\theta}}$$

2. Find the turning points on the curve $x = t$ and $y = t^3 - 3t$ and determine their nature

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$ ← use to find s.p's

$$= 3t^2 - 3$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(3t^2 - 3) \cdot \frac{dt}{dx}$$ ← use to determine nature

$$= 6t$$

Now s.p's when $\frac{dy}{dx} = 0$

$$3(t^2 - 1) = 0$$
$$3(t-1)(t+1) = 0$$
$$t = 1, -1$$

when $t = 1$, $\frac{d^2y}{dx^2} = 6 > 0$ min TP

when $t = -1$ $\frac{d^2y}{dx^2} = -6 < 0$ max tp

Summary

when $t = 1$, $x = 1$, $y = -2$

$\Rightarrow$ min TP at $(1, -2)$

when $t = -1$, $x = -1$, $y = 2$

$\Rightarrow$ max TP at $(-1, 2)$

Exercise **Sheet, Exercise 6, ALL questions;**
MiA AH2, p44, Ex.8A, Q1,2,3,4a,5;
MiA AH2, p45 Ex8B select

DO rectilinear reminder first

Parametric Differentiation – Motion in a Plane

Recall from **Higher** that the motion of an object moving in a **straight line** can be described as :-

displacement, $s(t)$ or x ;

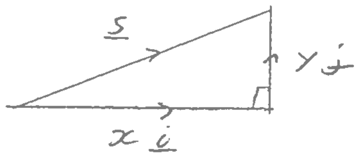
velocity, v ; $\quad v(t) = s'(t) = \dfrac{ds}{dt} = \dfrac{dx}{dt} = \dot{x}$

acceleration, a ; $\quad a(t) = v'(t) = \dfrac{dv}{dt} = s''(t) = \ddot{x}$

This is adapted for **motion in a plane** such that :-

displacement, $\quad s(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where **i** and **j** are the conventional unit vectors in the x and y directions.

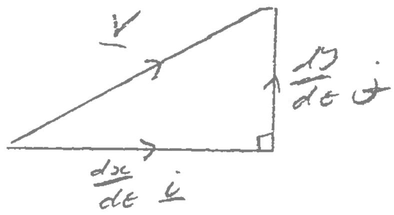
the magnitude of displacement, $|s(t)| = \sqrt{\left((x(t))^2 + (y(t))^2\right)}$



velocity, $\quad v(t) = s'(t) = \dfrac{ds}{dt} = x'(t)\mathbf{i} + y'(t)\mathbf{j} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

the magnitude of velocity is the speed:-

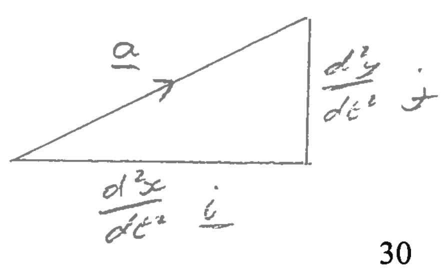
$$\text{speed, } |v(t)| = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{\left[(x'(t))^2 + (y'(t))^2\right]}$$



acceleration, $\quad a(t) = v'(t) = \dfrac{dv}{dt} = s''(t) = \dfrac{d^2s}{dt^2} = x''(t)\mathbf{i} + y''(t)\mathbf{j} = \dfrac{d^2x}{dt^2}\mathbf{i} + \dfrac{d^2y}{dt^2}\mathbf{j} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

the magnitude of acceleration can be written as $|a(t)|$ and calculated as shown below:

$$|a(t)| = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right) + \left(\frac{d^2y}{dt^2}\right)\right]} = \sqrt{\left[(x''(t))^2 + (y''(t))^2\right]}$$



7. **Rectilinear Motion** (Motion in a straight line)

Assume a body's motion is only in a straight line, say along the x axis. Naturally its displacement or position (s or x) will vary with time (t).

The rate of change of position of a body with respect to time is the body's **velocity**, and may be written:-

$$v(t) \quad = \quad \frac{ds}{dt} \quad \text{or} \quad \frac{dx}{dt} \quad \text{or} \quad \dot{x}$$

The rate of change of a body's velocity with respect to time is the body's **acceleration** and may be written:-

$$a(t) \quad = \quad \frac{dv}{dt} \quad \text{or} \quad v'(t) \quad \text{or} \quad \frac{d^2x}{dt^2} \quad \text{or} \quad \frac{d^2s}{dt^2} \quad \text{or} \quad \ddot{x}$$

Example

An object is moving in a straight line such that after t seconds motion its position relative to a fixed point, O, is given by :-

$$x = 9t + 3t^2 - t^3 , \qquad \text{where } x \text{ is in metres}$$

(a) Find the initial velocity and acceleration of the object

(b) Find the velocity and acceleration after 4 seconds. Comment on the acceleration.

(c) At what time does the object come to rest ? What is its position then ?

(d) When is the object not accelerating ?

(a) $\dot{x} = 9 + 6t - 3t^2$, $\qquad \ddot{x} = 6 - 6t$

$\dot{x}(0) = 9 \ ms^{-1}$ $\qquad \ddot{x}(0) = 6 ms^{-2}$

b) $\dot{x} = 9 + 6t - 3t^2$ $\qquad \ddot{x} = 6 - 6t$

$\dot{x}(4) = 9 + 24 - 3\times16$ $\qquad \ddot{x}(4) = 6 - 6\times4$

$= 33 - 48$ $\qquad = 6 - 24$

$= -15 \ ms^{-1}$ $\qquad = -18 \ ms^{-1}$

(going backwards) $\qquad$ (decelerating)

c) At what time does $v = 0$?

$\dot{x} = 9 + 6t - 3t^2$, $\dot{x} = 0$

$0 = 9 + 6t - 3t^2$

$0 = 3(3 + 2t - t^2)$ $\begin{matrix} 3 & -t \\ 1 & t \end{matrix}$

$0 = 3(3 - t)(1 + t)$

$t = 3, \cancel{-1}_{N/A}$

$x = 9t + 3t^2 - t^3, t = 3$

$= 27 + 27 - 27$

$= 27$

The object comes to rest after 3 seconds and is 27m from O

d) When does $a = 0$?

$\ddot{x} = 6 - 6t$, $\ddot{x} = 0$

$0 = 6 - 6t$

$6t = 6$

$t = 1$

The object is not accelerating at $t = 1$ second.

p50-53 Ex1 Q1 b, c, d, f

Q2 b, c, d, f

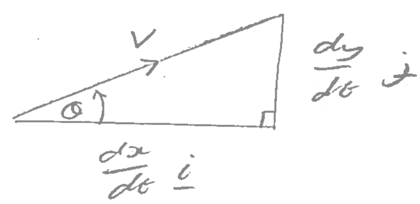
Q3, 5, 8, 10, 11, 12.

Direction of Motion

The **direction of velocity and acceleration** at any given time, t, is referenced to the **x direction** and is given by :-

$$\tan\theta = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \qquad \text{or} \qquad \tan\theta = \frac{d^2y/dt^2}{d^2x/dt^2} = \frac{y''(t)}{x''(t)}$$

eg.



Example

The flight of a golf ball, t seconds after being hit, is given by the parametric equations:

$$x = 10t, \qquad y = 30t - 5t^2$$

Calculate:-

(a) The position of the golf ball after 2 seconds

(b) The speed of the golf ball when it hits the ground, and the direction of the velocity at this instant.

(c) The magnitude of the acceleration of the golf ball when $t = 5$ seconds

(a) $x(t) = 10t \quad y(t) = 30t - 5t^2$

$x(2) = 20 \quad y(2) = 60 - 20$

$= 40$

$\Rightarrow \underline{\underline{(20, 40)}}$

(b) The ball hits the ground when

$y = 0$

$5t(6 - t) = 0$

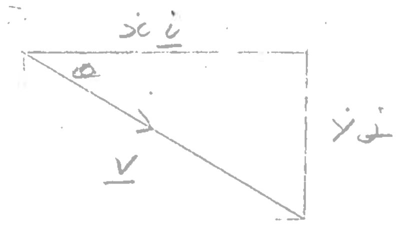
$t = 0$ or $t = 6$

↑ initial position

↑ ball landing

Exercise **MiA AH2; page 50/51; Exercise 1; Q1, 2, 4, 5, 6, 7, 9**

(b)



$x(t) = 10t \quad y(t) = 30t - 5t^2$

$\dot{x}(t) = 10 \quad \dot{y}(t) = 30 - 10t$

$\dot{x}(6) = 10 \quad \dot{y}(6) = -30$

$$\text{Speed } |V| = \sqrt{\dot{x}(6)^2 + \dot{y}(6)^2}$$

$$= \sqrt{10^2 + (-30)^2}$$

$$= \sqrt{100 + 900}$$

$$= \sqrt{1000}$$

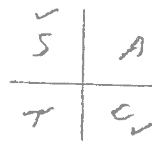
$$= \underline{\underline{31.6 \text{ ms}^{-1}}} \ (1 \text{ dp})$$

For direction, $\tan\theta = \dfrac{\dot{y}(6)}{\dot{x}(6)}$

$$= \frac{-30}{10}$$

$$= -3$$

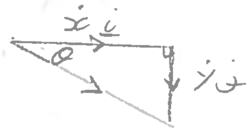
Decide which quadrant



r.a. = 71.

Since $\dot{y}(6) < 0$ and $\dot{x}(6) > 0$

It must be 4^{th} quadrant



Hence $\underline{\underline{\theta = -71.6^\circ}}$

(c) $\dot{x}(t) = 10 \quad \dot{y}(t) = 30 - 10t$

$\ddot{x}(t) = 0 \quad \ddot{y}(t) = -10$

$\ddot{x}(5) = 0 \quad \ddot{y}(5) = -10$

$$|a| = \sqrt{\ddot{x}(5)^2 + \ddot{y}(5)^2}$$

$$= \sqrt{0^2 + (-10)^2}$$

$$= \sqrt{100}$$

$$= \underline{\underline{10 \text{ ms}^{-2}}}$$