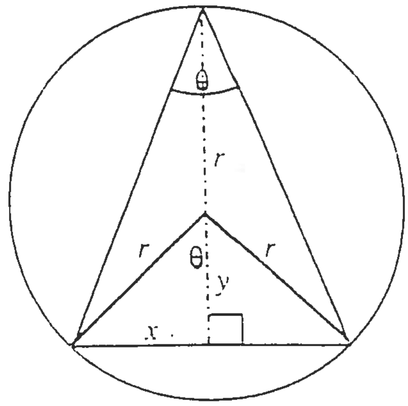
Optimisation

Differentiation can be used to analyse problem sand help find "optimum" values of systems, as long as they can be modelled satisfactorily.

Example

The diagram shows an isosceles triangle inscribed in a circle, radius, r.



(a) Show that the area of the triangle is given by:-

$$A = r^2 \sin\theta(1 + \cos\theta)$$

where θ is the angle between the equal sides.

(b) Find the maximum possible area of the triangle.

(a) Area $= \frac{1}{2} bh$

$= \frac{1}{2}(2x)(y+r)$

$= x(y+r)$

find x and y in terms of θ or r.

$\sin\theta = \frac{x}{r}$ $\quad$ $\cos\theta = \frac{y}{r}$

$x = r\sin\theta$ $\quad$ $y = r\cos\theta$

$A(\theta) = r\sin\theta(r\cos\theta + r)$

$= r^2 \sin\theta(1 + \cos\theta)$

(b) $A'(\theta) = r^2[\cos\theta(1+\cos\theta) + \sin\theta \cdot -\sin\theta]$ $\quad$ Product rule

$= r^2(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= r^2(\cos\theta + \cos 2\theta)$

$A(\theta)$ is continuous and $A'(\theta)$ exists for the interval $(0, \pi)$

$\Rightarrow$ Consider critical points where $A'(\theta) = 0$.

S.P.s when $A'(\theta) = 0$

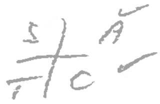
$$r^2(\cos\theta + \cos 2\theta) = 0$$

$$\cos\theta + \cos 2\theta = 0 \qquad (r \neq 0)$$

$$\cos\theta + 2\cos^2\theta - 1 = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$



$\Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ N/A (since $0 < \theta < \pi$) or $\theta = \pi$ N/A

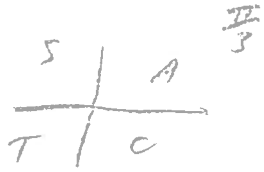
or use a nature table!

For nature of S.P. at $\theta = \frac{\pi}{3}$, find $A''(\theta)$

$$A'(\theta) = r^2(2\cos^2\theta + \cos\theta - 1)$$

$$A''(\theta) = r^2(4\cos\theta(-\sin\theta) - \sin\theta)$$

$$= -r^2\sin\theta(4\cos\theta + 1)$$



$$= -r^2\left(\frac{\sqrt{3}}{2}\right)(3)$$

$$= -ve$$

concave down

$\Rightarrow$ Maximum (local) when $\theta = \frac{\pi}{3}$

(b) For max Area, $A(\theta) = r^2\sin\theta(1 + \cos\theta)$

$$A\left(\frac{\pi}{3}\right) = r^2 \sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right)$$

$$= r^2\left(\frac{\sqrt{3}}{2}\right)\left(1 + \frac{1}{2}\right)$$

$$= r^2\left(\frac{3\sqrt{3}}{4}\right) \text{ units}^2$$

Exercise:- Do <u>Exercise 9</u> (below), then MiA AH 1, p63, Q 1, 2, 4, 9

Exercise 9

1. Four squares each of side s cm are cut from the corners of a metal square of side 16 cm. The metal is then bent to make an open topped tray of volume, V cm^3.

   (a) Prove that $V = 4s^3 - 64s^2 + 256s$.

   (b) Find the value of s which makes the volume a maximum.

2. A sector of a circle with radius r cm has an area of 16 cm^2.

   (a) Show that the perimeter P cm of the sector is given by

   $$P(r) = 2(r + \frac{16}{r})$$

   (b) Find the minimum value of P.

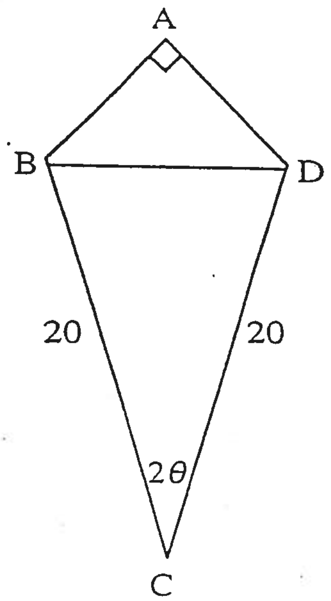
3. A cylindrical tank has a radius of r metres and a height of h metres. The sum of the radius and the height is 2 metres.

   (a) Prove that the volume, in m^3, is given by

   $$V = \pi r^2(2 - r)$$

   (b) Find the maximum volume.

4. ABCD is a kite which has AC as its axis of symmetry. Angle BAD is right angled and BC and DC are 20 cm.



   (a) Show that the area of triangle BCD is given by the expression $200 \sin 2\theta$ and find an expression for BD^2.

   (b) Use this expression for BD^2 to show that the area of triangle BAD is given by the expression $200 - 200 \cos 2\theta$ and hence show that the area of the kite is given by the expression $A(\theta) = 200(1 - \cos 2\theta + \sin 2\theta)$

   (c) Find the value of θ which makes the area a maximum and find this maximum area.

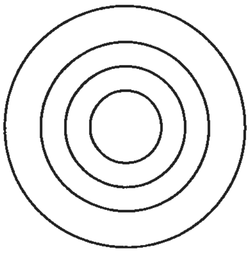
Exercise 9 Page 25 - 26

1.	(a)	Proof	(b)	$s = 8/3$ cm
2.	(a)	Proof	(b)	P = 16 cm
3.	(a)	Proof	(b)	$V = 32\pi/27$ cm3
4.	(a)	Proof	(b)	Proof
	(c)	$\theta = \overset{3\pi/8}{\cancel{8\pi/3}}$, $A = 200(1 + \sqrt{2})$		

Unit 2, Outcome 2 – Further Differentiation

9. Related Rate Problems – Application of the Chain Rule

When a pebble is dropped into a pool of water a series of concentric circles is produced. Both the radius and area of each circle vary with time, and naturally the area of each circle varies with the radius. The *rate of change of radius* with respect to



time, $\frac{dr}{dt}$, *rate of change of area* with respect to time, $\frac{dA}{dt}$, *rate of change of area* with respect to radius, $\frac{dA}{dr}$, are all related and relationships may made using the chain rule:

e.g. $$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

Many such relationships can be found and are of value when solving problems involving rates.

Examples

1. A stone is dropped into a pool of water, creating circular ripples. Find the rate at which the area of the outer circle is increasing (in cm²/s), at the point where the radius is 4 cm.

   The rate at which the radius is changing is $\frac{3}{2}$ cm/s.

ie. Find $\frac{dA}{dt}$ when $r=4$ as $\frac{dr}{dt} = \frac{3}{2}$

NOW $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$, $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$= 2\pi r \cdot \frac{3}{2}$

$= 3\pi r$

when $r=4$, $\frac{dA}{dt} = 3\pi(4)$

$= 12\pi$ cm²/sec.

2. A spherical balloon is blown up such that its volume increases at the constant rate of 8 cm^3/s.
At what rate is the radius increasing at the instant the radius is 4 cm ?

$\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$. Find $\frac{dr}{dt}$ when $r = 4$

$V = \frac{4}{3}\pi r^3$ (sphere)

$\frac{dV}{dr} = 4\pi r^2$

Hence $\frac{dr}{dV} = \frac{1}{4\pi r^2}$

Now $\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$

$= 8 \cdot \frac{1}{4\pi r^2}$

$= \frac{2}{\pi r^2}$

when $r = 4$, $\frac{dr}{dt} = \frac{2}{\pi (4)^2}$

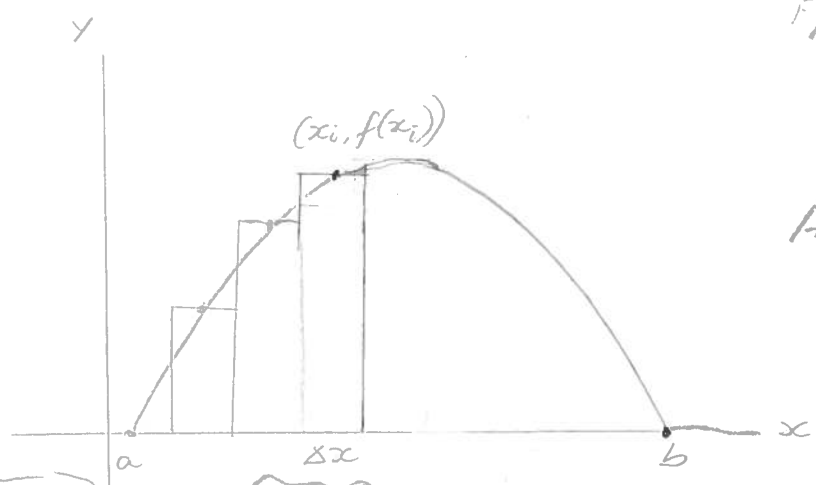
$= \frac{2}{16\pi}$

$= \frac{1}{8\pi}$ cm/sec.

Exercise **MiA AH2; page 53/54; Exercise 2A – ALL questions;**
page 54/55; Exercise 2B – Q1, 2, 3, 4, 5, 8, 9, 10

Areas Between the Curve and the x-axis

Area$_\square$ = $f(x)\,\Delta x$



Area = $\lim\limits_{\Delta x \to 0} \sum\limits_{x=a}^{b} f(x)\cdot\Delta x$

$= \int_a^b f(x)\,dx$

by definition

$\Delta x = \frac{b-a}{n}$ for n rectangles

As $n \to \infty$, $\Delta x \to 0$

$$\text{Area} = \int_a^b f(x)dx$$

Note Always sketch graph first.
Areas below x-axis give negative answer.
Work out areas above and below x-axis separately.

Examples

1. Calculate the area bounded by $y = x^2 - 4x$ and the x-axis.

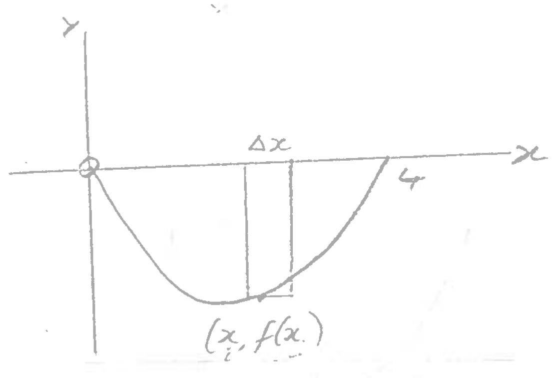
For roots

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

Note $f(x) < 0$ but area cannot be negative, so



Area = $\lim\limits_{\Delta x \to 0} \sum\limits_{x=a}^{b} -f(x)\cdot\Delta x$

$= \int_0^4 -f(x)\,dx$

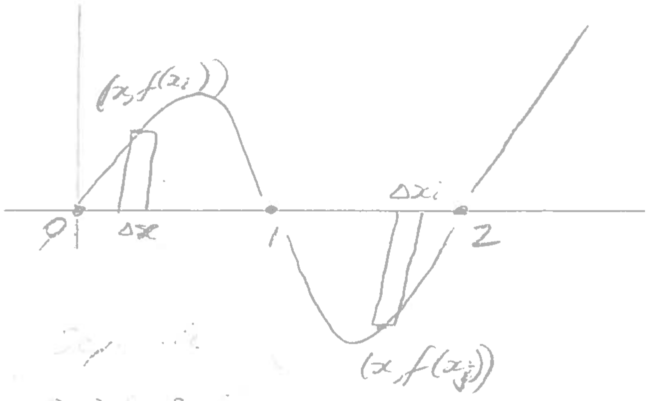
$= -\int_0^4 f(x)\,dx$

etc.

2. Find the area between the curve y = x(x – 1)(x – 2) and the x-axis.

Sketch it !!

Roots at $x = 0, 1, 2$



$\text{Area}_i = f(x_i) \cdot \Delta x$ for $[0, 1]$
$\text{Area}_j = -f(x_j) \Delta x$ for $[1, 2]$

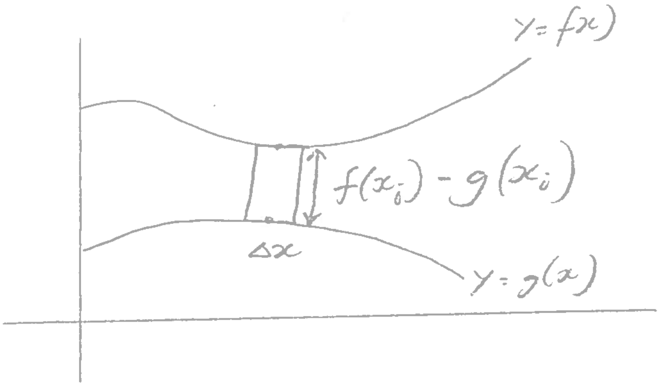
So sum the areas separately for above and below.

OR

$$\text{Area} = \underbrace{\int_0^1 f(x)\,dx}_{\text{above}} - \underbrace{\int_1^2 f(x)\,dx}_{\text{below}}$$

Maths in Action Book 1 Page 88 Exercise 10A Questions 3 , 4

Area Between Two Curves



$\text{Area}_i = (f(x_i) - g(x_i)) \cdot \Delta x$

$$\text{Area} = \int_a^b (upper - lower)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx$$

Example

Calculate the area between the curves y = $x^2 + 6$ and y = $12 + 4x - x^2$.

Sketch

P.O.I when

$x^2 + 6 = 12 + 4x - x^2$

$2x^2 - 4x - 6 = 0$

$2(x^2 - 2x - 3) = 0$

$2(x+1)(x-3) = 0$

$x = -1, 3$

$\text{Area} = \int \text{upper} - \text{lower}\,dx$

$= \int_{-1}^{3} 12 + 4x - x^2 - (x^2 + 6)\,dx$

$= \int_{-1}^{3} 6 + 4x - 2x^2\,dx$

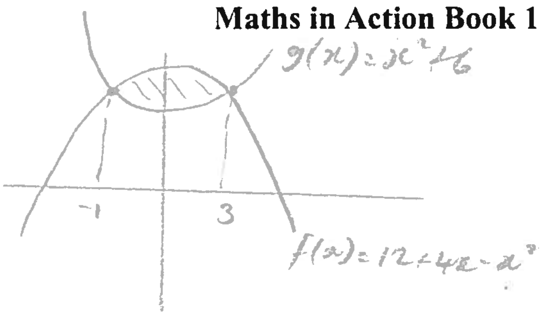
$= \left[6x + 2x^2 - \frac{2x^3}{3}\right]_{-1}^{3}$

$= (18 + 18 - 18) - (-6 + 2 + \frac{2}{3})$

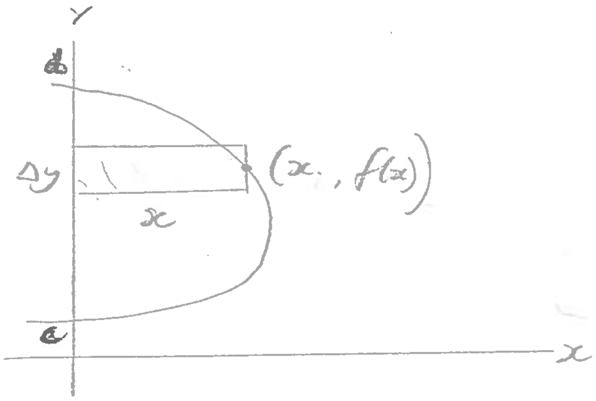
$= 18 - (-3\frac{1}{3})$

$= 21\frac{1}{3}$ units^2

Maths in Action Book 1 Page 90 Exercise 10b Questions 4 , 5 , 8



Area Between the Curve and y-axis



$$\text{Area}_{\square} = x \cdot \Delta y$$

$$\text{Area} = \lim_{\Delta y \to 0} \sum_{y=c}^{d} x \cdot \Delta y$$

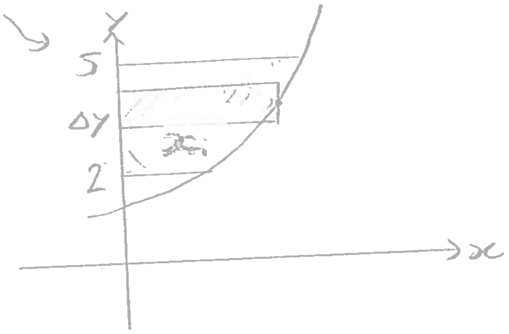
$$= \int_c^d x \, dy$$

where x is a function of y.

$$\text{Area} = \int_c^d g(y)dy$$

limits for y!

Example $y = x^2 + 1$



$$\text{Area} = \int_2^5 x \, dy$$

Here $y = x^2 + 1$

$y - 1 = x^2$

$x = +\sqrt{y-1}$

positive value for x in this part of graph.

$$= \int_2^5 (y-1)^{\frac{1}{2}} dy$$

$$= \left[\frac{(y-1)^{\frac{3}{2}}}{\frac{3}{2} \cdot 1} \right]_2^5$$

$$= \left[\tfrac{2}{3}(y-1)^{\frac{3}{2}} \right]_2^5$$

$$= \tfrac{2}{3}(4)^{\frac{3}{2}} - \tfrac{2}{3}(1)^{\frac{3}{2}}$$

$$= \tfrac{2}{3} \times 8 - \tfrac{2}{3}$$

$$= \tfrac{14}{3} \text{ square units.}$$

NOTE Areas to the left of Y axis would give negative value but correct magnitude.

Tips for Success….

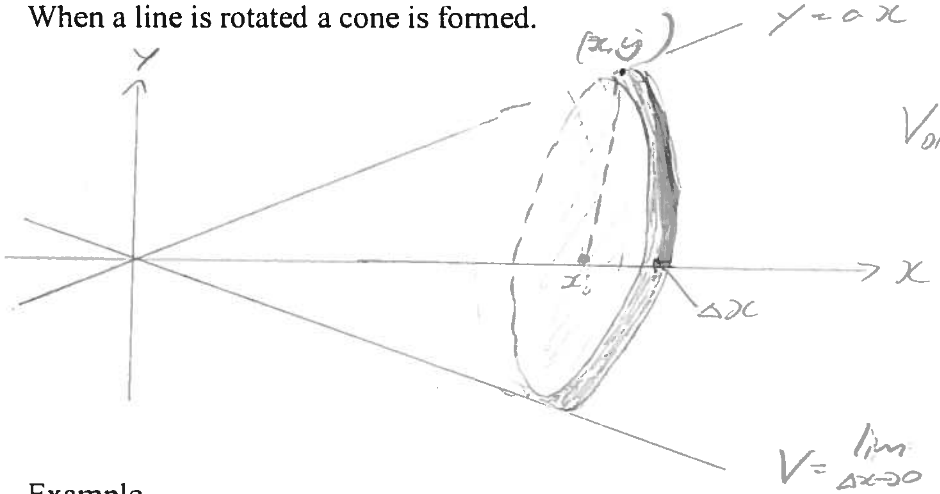
- Calculating areas <u>always</u> use a sketch
- check if area (i) above / below for x axis
  (ii) left / right for y axis
  (iii) upper / lower curve for areas between.

Remember
- for area between curve + x axis everything in terms of x
- for area between curve + y axis everything in terms of y

Worksheet Q15 – 18

Volumes of Revolution

Volumes of revolution are formed when a curve is rotated about the x-axis (or y-axis).
When a line is rotated a cone is formed.



$V_{DISC} = \pi r^2 h$

$= (\pi y^2)\, \Delta x$

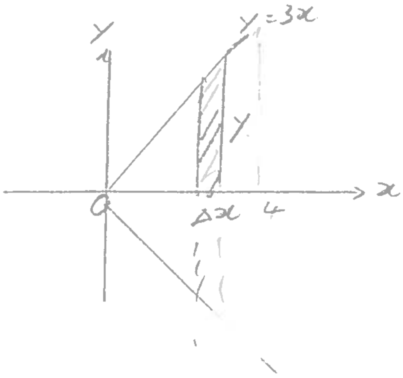
For n discs,

$V = \lim_{\Delta x \to 0} \sum_a^b \pi y_i^2 \, \Delta x$

$\Rightarrow V = \int_a^b \pi y^2 \, dx$

Example

If $y = 3x$, between $x = 0$ and $x = 4$



$V = \int_0^4 \pi y^2 \, dx$

$= \int_0^4 (3x)^2 \pi \, dx$

$= 9\pi \left[\frac{x^3}{3}\right]_0^4$

$= 9\pi \left(\frac{4^3}{3} - 0\right)$

$= 192\pi$ units^3

In general,

limits for x

Using the x-axis $\qquad V = \int_a^b \pi y^2 dx$

in terms of x

limits for y

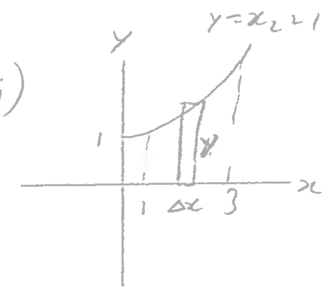
Using the y-axis $\qquad V = \int_c^d \pi x^2 dy$

in terms of y

Example

Find the volume of revolution obtained between x = 1 and x = 3 when the curve $y = x^2 + 1$ is rotated (i) about the x-axis
(ii) about the y-axis.

(i)



$V_{D/x} = \pi y^2 \Delta x$

$V = \int_1^3 \pi y^2 dx$

$= \pi \int_1^3 (x^2+1)^2 dx$

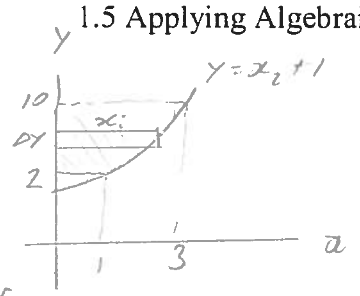
$= \pi \int_1^3 x^4 + 2x^2 + 1 \, dx$

$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_1^3$

$= \pi \left[\left(\frac{3^5}{5} + \frac{2(5)^3}{3} + 5\right) - \left(\frac{1^5}{5} + \frac{2(1)^3}{3} + 1\right)\right]$

$= \frac{1016}{15}\pi$

(ii)



limits for y

$y = x^2 + 1$

$\Rightarrow x^2 = y - 1$

$V_{DISC} = \pi x_i^2 \Delta y$

$V = \int_2^{10} \pi x^2 \, dy$

$= \pi \int_2^{10} (y-1) \, dy$

$= \pi \left[\frac{y^2}{2} - y \right]_2^{10}$

$= \pi \left[\left(\frac{10^2}{2} - 10 \right) - \left(\frac{2^2}{2} - 2 \right) \right]$

$= \pi [40 - 0]$

$= \underline{40\pi \text{ units}^3}$

Tips for Success….

Remember

$V = \int_a^b \pi y^2 \, dx$ replace with x's

about x axis

$\Rightarrow$ all in x

$V = \int_c^d \pi x^2 \, dy$ replace with y's

about y axis

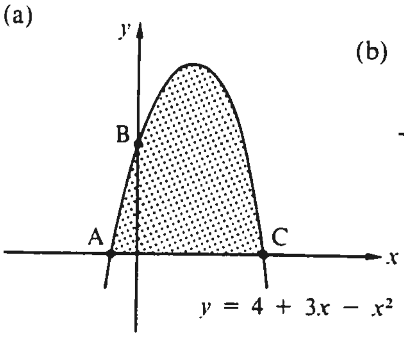
all in y's

Maths in Action Book 1 Page 88 Exercise 10A Questions 5 , 14
Page 93 Review Question 11

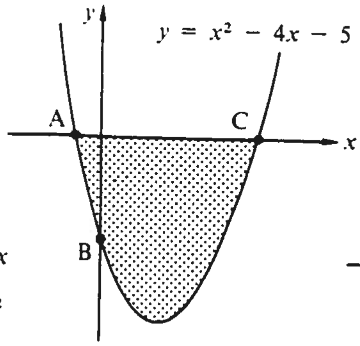
Integration Areas About x-axes and y-axis

2. For each of the following, find the coordinate of points A, B and C and find the shaded area.

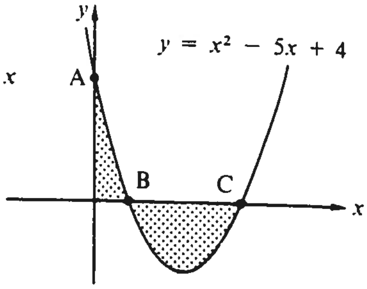
(a)



(b)



(c)

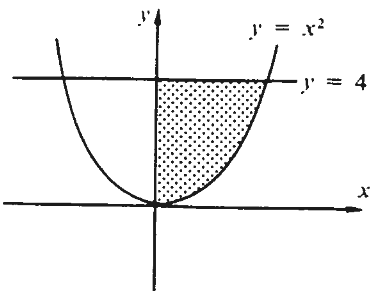


3. State why each of the following has no meaning

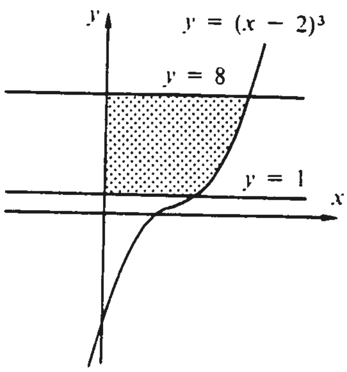
(a) $\int_{-3}^{4} \frac{1}{x} dx$ (b) $\int_{0}^{4} \frac{1}{x^2} dx$ (c) $\int_{-3}^{3} \frac{1}{x-1} dx$ (d) $\int_{-5}^{0} \frac{1}{x^2-1} dx$ (e) $\int_{-2}^{2} \sqrt{x}\, dx$.

4. Find the area between the line $y = 2x + 3$ and the x-axis from $x = 4$ to $x = 6$.
5. Find the area between the curve $y = x^3$ and the x-axis from $x = 1$ to $x = 2$.
6. Find the area enclosed by the lines $y = x^2 + 2$, the x-axis, $x = 1$ and $x = 3$.
7. Find the area between the curve $y = 10 + 3x - x^2$ and the x-axis from $x = -1$ to $x = 2$.
8. Find the area enclosed by $y = 3 + 2x - x^2$ and the x-axis.
9. Find the area enclosed by $y = x^2 - 6x$ and the x-axis.
10. Find the area between $y = 1 + \frac{4}{x^2}$ and the x-axis from $x = 1$ to $x = 2$.
11. Find the area between the curve $y = x^2 - 6x + 5$ and the x-axis from $x = 0$ to $x = 5$.
12. Find the area between the curve $y = 4 - x^2$ and the x-axis from $x = 0$ to $x = 3$.
13. Find the total area enclosed between $y = (x^2 - 1)(x - 3)$ and the x-axis.
14. Find the total area between the curve $y = \frac{4}{x^2} - 1$ and the x-axis from $x = 1$ to $x = 3$.
15. Using area $= \int_{y=a}^{y=b} x\, dy$, find the following shaded areas:

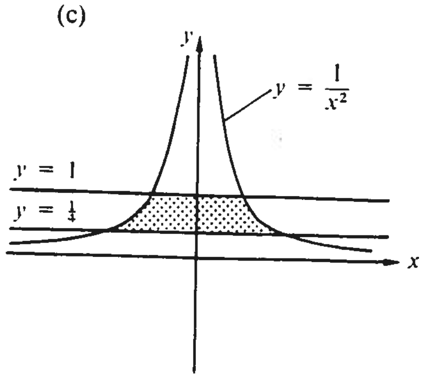
(a)



(b)



(c)



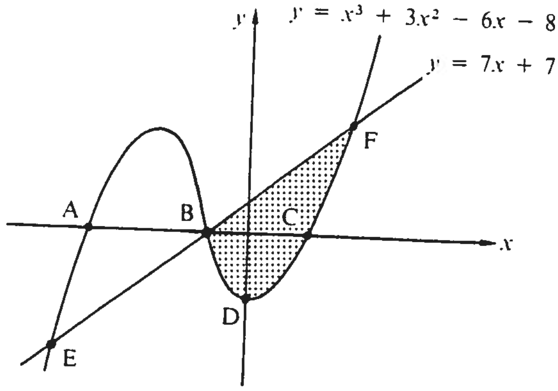
16. Find the area enclosed between the curve $y^2 = 4 - x$ and the y-axis.

17. Find the area enclosed by the curve $(y - 1)^2 = x$, the y-axis and the line $y = 3$.

18. Find the area enclosed between the curve $y = x^2$ and the straight line $y = x$.

19. The line $y = x + 8$ cuts the curve $y = 12 + x - x^2$ at two points A and B (B being in the first quadrant). Find the coordinates of A and B and find the area enclosed between the curve $y = 12 + x - x^2$ and the straight line AB.

20. The sketch graph shows the lines $y = 7x + 7$ and $y = x^3 + 3x^2 - 6x - 8$. Find the coordinates of the points A, B, C, D, E and F and find the shaded area.



21. Find the area enclosed between the curves $y = x^2 + 6$ and $y = 12 + 4x - x^2$.

22. Find the area enclosed between the curves $y = x^2 - 4x$ and $y = 6x - x^2$.

Answers

2. (a) A(−1, 0), B(0, 4), C(4, 0), $20\frac{5}{6}$ sq. units
   (b) A(−1, 0), B(0, −5), C(5, 0), 36 sq. units
   (c) A(0, 4), B(1, 0), C(4, 0), $6\frac{1}{3}$ sq. units

3. (a) $\frac{1}{x}$ not defined for $x = 0$, (b) $\frac{1}{x^2}$ not defined for $x = 0$,
   (c) $\frac{1}{x - 1}$ not defined for $x = 1$, (d) $\frac{1}{x^2 - 1}$ not defined for $x = -1$
   (e) $\sqrt{x}$ not defined for $x < 0$

4. 26 sq. units
5. $3\frac{3}{4}$ sq. units
6. $12\frac{2}{3}$ sq. units
7. $31\frac{1}{2}$ sq. units
8. $10\frac{2}{3}$ sq. units
9. 36 sq. units
10. 3 sq. units
11. 13 sq. units
12. $7\frac{2}{3}$ sq. units
13. 8 sq. units
14. $1\frac{1}{3}$ sq. units
15. (a) $5\frac{1}{4}$ sq. units (b) $25\frac{1}{4}$ sq. units (c) 2 sq. units
16. $10\frac{2}{3}$ sq. units
17. $2\frac{2}{3}$ sq. units
18. $\frac{1}{6}$ sq. units
19. A(−2, 6), B(2, 10), $10\frac{2}{3}$ sq. units
20. A(−4, 0), B(−1, 0), C(2, 0), D(0, −8), E(−5, −28), F(3, 28), 64 sq. units
21. $21\frac{1}{3}$ sq. units
22. $41\frac{2}{3}$ sq. units

Volume of Revolution

Find the volumes of the solids formed when each of the areas of questions **1** to **10** perform one revolution about the x-axis.

1. The area enclosed by the curve $y = x^2$, the x-axis and the line $x = 2$.
2. The area between the line $y = x + 1$ and the x-axis from $x = 1$ to $x = 3$.
3. The area between the line $y = 3x + 2$ and the x-axis from $x = 0$ to $x = 1$.
4. The area enclosed by the curve $y = x^3$, the x-axis and the line $x = 2$.
5. The area enclosed by the curve $y = x^2 + 1$, the x-axis, $x = -1$ and $x = 1$.
6. The area enclosed by the curve $y = x^3 + 1$, the x-axis and the line $x = 1$.
7. The area enclosed by the curve $y = 4x - x^2$ and the x-axis.
8. The area enclosed by the curve $y = x^2 - x^3$ and the x-axis.
9. The area enclosed by the curve $y = x - \frac{1}{x}$ the x-axis and the line $x = 2$.
10. The area enclosed between the curve $y = 4x - x^2$ and the line $y = 2x$.
11. Find the equation of the straight line joining the origin to the point with coordinates (h, r). Hence find a formula for the volume of a right circular cone of base radius r and height h.

Find the volumes of the solids formed when each of the areas of questions **12** to **14** perform one revolution about the y-axis.

12. The area lying in the first quadrant and bounded by the curve $y = x^2$, the y-axis and the line $y = 4$.
13. The area lying in the first quadrant and bounded by the curve $y = 2x^2 + 1$, the y-axis and the lines $y = 2$ and $y = 5$.
14. The area lying in the first quadrant and bounded by the y-axis, the curve $y = x^3$ and the line $y = 3x + 2$.

Answers.

1. $\frac{32}{5}\pi$ cu. units
2. $\frac{56}{3}\pi$ cu. units
3. 13π cu. units
4. $\frac{128}{7}\pi$ cu. units
5. $\frac{56}{15}\pi$ cu. units
6. $\frac{16}{7}\pi$ cu. units
7. $\frac{512}{15}\pi$ cu. units
8. $\frac{1}{105}\pi$ cu. units
9. $\frac{5}{6}\pi$ cu. units
10. $\frac{32}{5}\pi$ cu. units
11. $y = \frac{rx}{h}$, $V = \frac{1}{3}\pi r^2 h$
12. 8π cu. units
13. $\frac{15}{4}\pi$ cu. units
14. $\frac{56}{5}\pi$ cu. units