

AH Maths 2015

$$\textcircled{1} \quad \left(\frac{x^2}{3} - \frac{2}{x} \right)^5$$

$$\begin{aligned}
 &= \left(\frac{x^4}{3} \right)^5 + 5 \left(\frac{x^4}{3} \right)^4 \left(-\frac{2}{x} \right)^1 \\
 &\quad + 10 \left(\frac{x^4}{3} \right)^3 \left(-\frac{2}{x} \right)^2 + 10 \left(\frac{x^4}{3} \right)^2 \left(-\frac{2}{x} \right)^3 + 5 \left(\frac{x^4}{3} \right) \left(-\frac{2}{x} \right)^4 \\
 &\quad + \left(-\frac{2}{x} \right)^5
 \end{aligned}$$

$$= \frac{x^{10}}{243} - \frac{10}{81}x^7 + \frac{40}{27}x^4 - \frac{80}{9}x + \frac{80}{3x^2} - \frac{32}{x^5}$$

1					
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

$$\textcircled{2} \quad (\text{a}) \quad y = \frac{5x+1}{x^2+2}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(x^2+2) \cdot 5 - (5x+1) \cdot 2x}{(x^2+2)^2} \\
 &= \frac{5x^2+10 - 10x^2-2x}{(x^2+2)^2} \\
 &= \frac{-5x^2-10-2x}{(x^2+2)^2} \\
 &= \frac{-(5x^2+2x+10)}{(x^2+2)^2}
 \end{aligned}$$

$$(\text{b}) \quad f(x) = e^{2x} \sin^2 3x$$

$$\begin{aligned}
 f'(x) &= e^{2x} \cdot 2 \sin 3x \cdot \cos 3x \cdot 3 \\
 &\quad + \sin^2 3x \cdot 2e^{2x} \\
 &= 6e^{2x} \sin 3x \cos 3x + 2e^{2x} \sin^2 3x \\
 &= 2e^{2x} \sin 3x (3 \cos 3x + \sin 3x)
 \end{aligned}$$

(2)

③

$$S_n = 320$$

$$U_n = 37$$

$$S_{20} = 320$$

$$U_{21} = 37$$

$$U_n = a + (n-1)d$$

$$37 = a + 20d \quad \dots \textcircled{1}$$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$320 = 10 (2a + 19d)$$

$$32 = 2a + 19d \quad \dots \textcircled{2}$$

$$\textcircled{1} \times 2$$

$$74 = 2a + 40d$$

②

$$32 = 2a + 19d$$

$$\text{subtract } \cancel{74} = 21d$$

$$d = 2 \Rightarrow a = -3$$

$$S_{10} = \frac{1}{2} (2a + (n-1)d) \quad n=10 \quad a=-3 \quad d=2$$

$$= 5 (-6 + 9 \times 2)$$

$$= 5 \times 12$$

$$= 60$$

④

$$x^4 + y^4 + 9x - 6y = 14$$

Differentiate

$$4x^3 + 4y^3 \frac{dy}{dx} + 9 - 6 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (4y^3 - 6) = -4x^3 - 9$$

$$\frac{dy}{dx} = -\frac{4x^3 + 9}{4y^3 - 6}$$

At point A(1, 2)

$$\frac{dy}{dx} = -\frac{4-9}{32-6}$$

$$= -\frac{13}{26}$$

$$= -\frac{1}{2}$$

Equation of tangent $y - b = m(x - a)$

$$y - 2 = -\frac{1}{2}(x-1)$$

$$2y - 4 = -x + 1$$

$$2y + x = 5$$

⑤

$$A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

Singular $\det A = 0$

$$p \left| \begin{matrix} p & 1 \\ -1 & 1 \end{matrix} \right| - 2 \left| \begin{matrix} 3 & 1 \\ 0 & -1 \end{matrix} \right| + 0 = 0$$

$$p(-p+1) - 2(-3-0) = 0$$

$$-p^2 + p + 6 = 0$$

$$p^2 - p - 6 = 0$$

$$(p-3)(p+2) = 0$$

$$p=3 \text{ or } p=-2$$

⑥

$$y = 3^{x^2}$$

Take logs

$$\ln y = \ln 3^{x^2}$$

$$\ln y = x^2 \ln 3$$

Differentiate

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x \ln 3$$

$$\frac{dy}{dx} = y \cdot 2x \ln 3$$

$$= (3^{x^2}) \cdot 2x \ln 3$$

⑦

Find gcd of (3066, 713)

$$3066 = 4 \times \underline{713} + \underline{214} \quad \text{--- ③}$$

$$713 = 3 \times \underline{214} + \underline{71} \quad \text{--- ②}$$

$$214 = 3 \times \underline{71} + \underline{1} \quad \text{--- ①}$$

$$1 = 71 \times \underline{1} + \underline{0}$$

$$\begin{aligned}\gcd(3066, 713) \\ = 1\end{aligned}$$

Working backwards

① gives $1 = 214 - 3 \times 71$

② gives $1 = 214 - 3 \times (713 - 3 \times 214)$
 $1 = 10 \times 214 - 3 \times 713$

③ gives $1 = 10 \times (3066 - 4 \times 713) - 3 \times 713$
 $1 = 10 \times 3066 - 43 \times 713$

i.e. $3066 \times 10 + 713 \times (-43) = 1$
 $p = 10 \quad q = -43.$

$$\textcircled{8} \quad x = \sqrt{t+1} \quad y = \cot t$$

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}(t+1)^{-\frac{1}{2}} \\ &= \frac{1}{2(t+1)^{\frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= -\csc^2 t \cdot 2(t+1)^{\frac{1}{2}} \\ &= -2(t+1)^{\frac{1}{2}} \csc^2 t\end{aligned}$$

$$\textcircled{9} \quad \text{Show } \binom{n+2}{3} - \binom{n}{3} = n^2$$

$$\begin{aligned}\text{LHS} &= \binom{n+2}{3} - \binom{n}{3} \\ &= \frac{(n+2)!}{3!(n+2-3)!} - \frac{n!}{3!(n-3)!} \\ &= \frac{(n+2)(n+1)n}{6} - \frac{n(n-1)(n-2)}{6} \\ &= \frac{n}{6} \{ n^2 + 3n + 2 - (n^2 - 3n + 2) \} \\ &= \frac{n}{6} \{ 6n \} \\ &= n^2 \quad \text{as required.}\end{aligned}$$

$$\textcircled{10} \quad \int_0^2 x^2 e^{4x} dx$$

$$\begin{aligned}&= \left[x^2 \cdot \frac{1}{4} e^{4x} \right]_0^2 - \frac{1}{4} \int_0^2 e^{4x} \cdot 2x dx \\ &= (e^8 - 0) - \frac{1}{2} \left\{ \left[x \cdot \frac{1}{4} e^{4x} \right]_0^2 - \frac{1}{4} \int_0^2 e^{4x} dx \right\} \\ &= e^8 - \frac{1}{2} \left\{ \left(\frac{1}{2} e^8 - 0 \right) - \frac{1}{4} \left[\frac{1}{4} e^{4x} \right]_0^2 \right\} \\ &= e^8 - \frac{1}{4} e^8 + \frac{1}{8} \left(\frac{1}{2} e^8 - \frac{1}{4} \right)\end{aligned}$$

$$= \frac{3}{4}e^8 + \frac{1}{32}e^8 - \frac{1}{32}$$

$$= \frac{25}{32}e^8 - \frac{1}{32}.$$

(11) $M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{. reflection in } y=x$$

(12) Let first odd number = $2n+1 \quad n \in \mathbb{N}$
 next odd number = $2n+3$

difference between squares

$$= (2n+3)^2 - (2n+1)^2$$

$$= (4n^2 + 12n + 9) - (4n^2 + 4n + 1)$$

$$= 4n^2 + 12n + 9 - 4n^2 - 4n - 1$$

$$= 8n + 8$$

= $8(n+1)$ which is divisible by 8

(B) (a) $z = x+iy$

$$z^2 = |z|^2 - 4$$

$$(x+iy)^2 = x^2 + y^2 - 4$$

$$x^2 + 2xyi - y^2 = x^2 + y^2 - 4$$

Equate real parts $x^2 - y^2 = x^2 + y^2 - 4$

$$-2y^2 = -4$$

$$y^2 = 2 \quad y = \pm \sqrt{2}$$

$$\text{Equate imaginary parts} \quad 2xy = 0$$

$$x = 0$$

$$z = \pm \sqrt{2}i$$

$$(b) \quad z^2 = i (|z|^2 - 4)$$

$$(x+iy)^2 = i(x^2 + y^2 - 4)$$

$$x^2 + 2xyi - y^2 = i(x^2 + y^2 - 4)$$

Equate real parts.

$$x^2 - y^2 = 0$$

$$x = \pm y$$

Equate imaginary parts

$$2xy = x^2 + y^2 - 4$$

$$\text{Put } y = x \quad 2x^2 = 2x^2 - 4 \quad \text{not possible.}$$

$$y = -x \quad -2x^2 = 2x^2 - 4$$

$$-4x^2 = -4$$

$$x = \pm 1$$

$$\text{Solution } z = \pm 1$$

$$(14) \quad g(x) = f(x) + f(-x) \quad h(x) = f(x) - f(-x)$$

$$\begin{aligned} g(-x) &= f(-x) + f(-(-x)) \\ &= f(-x) + f(x) \\ &= f(x) + f(-x) \\ &= g(x) \end{aligned}$$

so $g(x)$ is an even function.

$$\begin{aligned} h(-x) &= f(-x) - f(-(-x)) \\ &= f(-x) - f(x) \\ &= -f(x) + f(-x) \end{aligned}$$

$$= - (f(x) - f(-x))$$

$$= -h(x)$$

so $h(x)$ is an odd function

$$\begin{aligned}g(x) + h(x) &= f(x) + f(-x) + f(x) - f(-x) \\&= 2f(x)\end{aligned}$$

$$\begin{aligned}F(x) &= \frac{1}{2}(g(x) + h(x)) \\&= \frac{1}{2}g(x) + \frac{1}{2}h(x)\end{aligned}$$

which is the sum of an odd and even function.

15 (a) L_1

$$\underline{\Gamma} = \underline{a} + t\underline{d}$$

$$\underline{\Gamma} = \left(\begin{matrix} 2 \\ 4 \end{matrix}\right) + t\left(\begin{matrix} 1 \\ -1 \end{matrix}\right)$$

L_2

$$\underline{\Gamma} = \underline{a} + s\underline{d}$$

$$\underline{\Gamma} = \left(\begin{matrix} -5 \\ 2 \end{matrix}\right) + s\left(\begin{matrix} -4 \\ 1 \end{matrix}\right)$$

(b) lines are not parallel.

Intersection

$$\left(\begin{matrix} 2 \\ 4 \end{matrix}\right) + t\left(\begin{matrix} 1 \\ -1 \end{matrix}\right) = \left(\begin{matrix} -5 \\ 2 \end{matrix}\right) + s\left(\begin{matrix} -4 \\ 1 \end{matrix}\right)$$

$$2+t = -5-4s$$

-- ①

$$4+2t = 2+4s$$

-- ②

$$1-t = 5+s$$

-- ③

① + ②

$$6+3t = -3$$

$$3t = -9$$

$$t = -3 \Rightarrow -1 = -5-4s$$

$$s = -1$$

Check in (3)

$$1 - (-3) = 5 + (-1) \quad \checkmark$$

So lines intersect.

Point of intersection with $t = -3$.

$$(-1, -2, 4)$$

(c) normal vector

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ -4 & 4 & 1 \end{vmatrix}$$

$$= \underline{i}(2+4) - \underline{j}(1-4) + \underline{k}(4+8)$$

$$= 6\underline{i} + 3\underline{j} + 12\underline{k}$$

$$\underline{n} \cdot \underline{r} = \underline{n} \cdot \underline{a}$$

$$\begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$6x + 3y + 12z = 12 + 12 + 12$$

$$6x + 3y + 12z = 36$$

(16)

$$\frac{dy}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x} \quad (\star)$$

auxiliary equation

$$m^2 + 2m + 10 = 0$$

$$m = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -2 \pm \frac{\sqrt{4 - 40}}{2}$$

$$= -2 \pm \frac{\sqrt{-36}}{2}$$

$$= -2 \pm \frac{6i}{2}$$

$$m_1 = -1 \pm 3i$$

$$C.F \quad y = Ae^{-x} \cos 3x + Be^{-x} \sin 3x$$

P.I Try $y = Ce^{2x}$

$$\frac{dy}{dx} = 2Ce^{2x}$$

$$\frac{d^2y}{dx^2} = 4Ce^{2x}$$

Substitute in (*)

$$4Ce^{2x} + 4Ce^{2x} + 10Ce^{2x} = 3e^{2x}$$

$$18Ce^{2x} = 3e^{2x}$$

$$18C = 3$$

$$C = \frac{1}{6}$$

$$P.I \quad y = \frac{1}{6}e^{2x}$$

General solution

$$y = Ae^{-x} \cos 3x + Be^{-x} \sin 3x + \frac{1}{6}e^{2x}$$

$$x=0 \quad y=1 \quad 1 = A + 0 + \frac{1}{6}$$

$$A = \frac{5}{6}$$

$$\begin{aligned} \frac{dy}{dx} &= Ae^{-x}(-3\sin 3x) + -Ae^{-x}\cos 3x \\ &\quad + Be^{-x}(3\cos 3x) - Be^{-x}\sin 3x + \frac{1}{3}e^{2x} \end{aligned}$$

$$x=0, \quad \frac{dy}{dx} = 0$$

$$0 = 0 - A + 3B - 0 + \frac{1}{3}$$

$$0 = -\frac{5}{6} + 3B + \frac{1}{3}$$

$$3B = \frac{3}{6} \quad B = \frac{1}{6}$$

$$\text{So } y = \frac{5}{6} e^{-x} \cos 3x + \frac{1}{6} e^{-x} \sin 3x + \frac{1}{6} e^{2x}$$

$$(17) \quad \int \frac{2x^3 - x - 1}{(x-3)(x^2+1)} dx$$

$$(x-3)(x^2+1) = x^3 + x - 3x^2 - 3 \\ = x^3 - 3x^2 + x - 3.$$

$$x^3 - 3x^2 + x - 3 \quad \begin{array}{r} 2 \\ \hline 2x^3 - x - 1 \\ 2x^3 - 3x^2 + 2x - 6 \\ \hline 6x^2 - 3x + 5 \end{array}$$

$$\text{So } \frac{2x^3 - x - 1}{(x-3)(x^2+1)} = 2 + \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}$$

Partiel Brücheins.

$$\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)} = \frac{A}{(x-3)} + \frac{Bx+C}{x^2+1}$$

$$6x^2 - 3x + 5 = A(x^2+1) + (Bx+C)(x-3)$$

$$\text{let } x=3 \quad 54 - 9 + 5 = A(10)$$

$$A = 5$$

$$(x^2) \quad 6 = A + B$$

$$B = 1$$

$$\text{constant} \quad 5 = A - 3C$$

$$C = 0$$

$$\text{So } \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)} = \frac{5}{x-3} + \frac{x}{x^2+1}$$

$$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)} dx = \int \left(2 + \frac{5}{x-3} + \frac{x}{x^2+1} \right) dx$$

$$= 2x + 5\ln|x-3| + \frac{1}{2}\ln|x^2+1| + C.$$

(18) $\frac{dV}{dt} = -k\sqrt{h}$, $k > 0$

(b) Cylinder $V = Ah$

so $\frac{dV}{dt} = A \frac{dh}{dt}$ (since A is constant)

$$\begin{aligned} \frac{dh}{dt} &= \frac{1}{A} \frac{dV}{dt} \\ &= \frac{1}{A} (-k\sqrt{h}) \\ &= -\frac{k}{A} \sqrt{h}. \end{aligned}$$

as required.

(b) $t=0$ $h=144\text{cm}$ $\frac{dh}{dt} = -0.3\text{cm/hr.}$

$$\frac{dh}{dt} = -\frac{k}{A} \sqrt{h}.$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{k}{A} dt$$

$$\int h^{-\frac{1}{2}} dh = -\frac{k}{A} \int dt$$

$$2h^{\frac{1}{2}} = -\frac{k}{A} t + C.$$

When $t=0$ $h=144$ $24=C$.

$h=144$, $t=0$ $\frac{dh}{dt} = -0.3$ $-0.3 = -\frac{k}{A} \cdot 12$.

$$k = \frac{1}{40} A$$

$$2h^{\frac{1}{2}} = -\frac{1}{40}t + 24$$

$$h^{\frac{1}{2}} = 12 - \frac{1}{80}t$$

$$h = \left(12 - \frac{1}{80}t\right)^2$$

(c) Tank empties when $h=0$

$$0 = \left(12 - \frac{1}{80}t\right)$$

$$\frac{1}{80}t = 12$$

$$t = 960 \text{ hours}$$

$$t = 40 \text{ days}$$

(d) $r = 20$ Find $\frac{dV}{dt}$

$$\begin{aligned}\frac{dV}{dt} &= -k\sqrt{h} \\ &= -\frac{1}{40}A \cdot \left(12 - \frac{1}{80}t\right) \\ &= -\frac{1}{40}\pi r^2 \left(12 - \frac{1}{80}t\right)\end{aligned}$$

$$t = 4 \text{ days} = \frac{4 \times 24}{96} \text{ hours} \quad r = 20$$

$$\begin{aligned}\frac{dV}{dt} &= -\frac{1}{40} \times \pi \times 20^2 \left(12 - \frac{1}{80} \times 96\right) \\ &= -108\pi\end{aligned}$$

Water is being applied at a rate of $108\pi \text{ cm}^3/\text{hr}$.