

Advanced Higher Maths 2014.

① (a) $f(x) = \frac{x^2 - 1}{x^2 + 1}$

$$\begin{aligned}
 f'(x) &= \frac{(x^2+1).2x - (x^2-1).2x}{(x^2+1)^2} \\
 &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} \\
 &= \frac{4x}{(x^2+1)^2}
 \end{aligned}$$

(b) $y = \tan^{-1}(3x^2)$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{1+(3x^2)^2} \cdot 6x \\
 &= \frac{6x}{1+9x^4}
 \end{aligned}$$

② $\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}$

General term

$$\begin{aligned}
 &{}^{10}C_r \left(\frac{2}{x}\right)^{10-r} \left(\frac{1}{4x^2}\right)^r \\
 &= {}^{10}C_r 2^{10-r} x^{-10} \left(\frac{1}{4}\right)^r x^{-2r} \\
 &= {}^{10}C_r \frac{2^{10-r}}{4^r} x^{-r-10} \\
 &= {}^{10}C_r 2^{10-3r} x^{-r-10}
 \end{aligned}$$

For term in $\frac{1}{x^{13}}$

$$-r-10 = -13$$

$$r = 3$$

$$\begin{aligned}
 \text{Term} &= {}^{10}C_3 2^1 x^{-13} \\
 &= \frac{240}{x^{13}}
 \end{aligned}$$

③

$$x+y+z=2$$

$$4x+3y-az=4$$

$$5x+6y+8z=11$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 3 & -a & 4 \\ 5 & 6 & 8 & 11 \end{array} \right)$$

$$\begin{array}{l} R_2 - 4R_1 \\ R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -a-4 & -4 \\ 0 & 1 & 3 & 1 \end{array} \right)$$

$$R_3 + R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -a-4 & -4 \\ 0 & 0 & -a-1 & -3 \end{array} \right)$$

$$\text{so } (-a-1)z = -3$$

$$z = \frac{3}{a+1}$$

For solution to exist $-a-1 \neq 0$

$$a \neq -1$$

$$\text{When } a = 2 \quad z = \frac{3}{a+1} = 1$$

$$\begin{array}{ll} R_2 \text{ gives } & -y - 6z = -4 \\ & -y - 6 = -4 \\ & -y = 2 \\ & y = -2 \end{array}$$

$$\begin{array}{ll} R_1 \text{ gives } & x + y + z = 2 \\ & x - 2 + 1 = 2 \\ & x = 3 \end{array}$$

$$(3, -2, 1)$$

$$\textcircled{4} \quad x = \ln(1+t^2) \quad y = \ln(1+2t^2)$$

$$\frac{dx}{dt} = \frac{1}{1+t^2} \cdot 2t \quad \frac{dy}{dt} = \frac{1}{1+2t^2} \cdot 4t$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \cdot \frac{dt}{dx} \\ &= \frac{4t}{1+2t^2} \cdot \frac{1+t^2}{2t} \\ &= \frac{2t(1+t^2)}{1+2t^2}\end{aligned}$$

$$\textcircled{5} \quad \underline{v} \times \underline{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= i(-1-12) - j(-2-3) + k(8-1)$$

$$= -13i + 5j + 7k$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -13 \\ 5 \\ 7 \end{pmatrix}$$

$$= -65 + 65 + 0$$

$$= 0$$

\underline{u} is perpendicular to $\underline{v} \times \underline{w}$

$$\textcircled{6} \quad e^y = x^3 \cos^2 x$$

Differentiate w.r.t. x

$$e^y \cdot \frac{dy}{dx} = x^3 \cdot 2 \cos x \cdot (-\sin x) + \cos^2 x \cdot 3x^2$$

$$x^3 \cos^2 x \frac{dy}{dx} = 3x^2 \cos^2 x - 2x^3 \cos x \sin x$$

$$\frac{dy}{dx} = \frac{3}{x} - 2 \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{3}{x} - 2\cos x \quad \text{as required with } a=3 \text{ and } b=-2$$

⑦ $A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ Prove $A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}$

$$\begin{aligned} n=1 \quad A^1 &= \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \\ &= A \quad \text{so result is true for } n=1 \end{aligned}$$

Assume result is true for $n=k$

$$\text{i.e. } A^k = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

Consider $n=k+1$

$$\begin{aligned} A^{k+1} &= A^k A^1 \\ &= \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 2a(2^k - 1) + a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 2 + 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

so true for $n=k+1$

Result is true for $n=1$ and if result is true for $n=k$ then it is also true for $n=k+1$, hence by induction the result is true $\forall n \geq 1$

$$⑧ \quad 4 \frac{dy}{dx^2} - 4 \frac{dy}{dx} + y = 0$$

auxiliary equation

$$4m^2 - 4m + 1 = 0$$

$$m = \frac{-(-4) \pm \sqrt{16 - 16}}{8}$$

$$(2m-1)(2m-1) = 0$$

$$m = \frac{1}{2}$$

$$\text{General Solution } y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$$

$$\text{When } x=0 \quad y=4 \quad 4=A$$

$$\text{When } x=0 \quad \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} + Be^{\frac{1}{2}x} + \frac{1}{2}Bxe^{\frac{1}{2}x}$$

$$3 = \frac{1}{2}A + B$$

$$B=1$$

$$\text{Solution } y = 4e^{\frac{1}{2}x} + xe^{\frac{1}{2}x}$$

$$⑨ \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \dots$$

$$= 1 - \frac{9x^2}{2} + \frac{81x^4}{24} + \dots$$

$$= 1 - \frac{9x^2}{2} + \frac{27}{8}x^4 + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned} e^{2x} &= 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots \end{aligned}$$

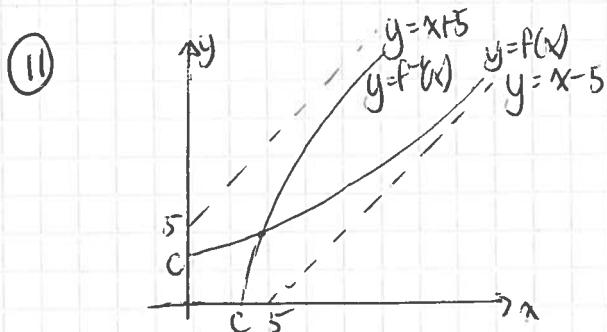
$$\begin{aligned}
 e^{2x} \cos 3x &= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots \right) \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^3 + \dots \right) \\
 &= 1 - \frac{9}{2}x^2 + 2x - 9x^3 + 2x^2 + \frac{4}{3}x^3 + \dots \\
 &= 1 + 2x - \frac{5}{2}x^2 - \frac{23}{3}x^3 + \dots
 \end{aligned}$$

(10) Equation of circle $(x-1)^2 + (y-0)^2 = 2^2$
 $(x-1)^2 + y^2 = 4 \quad \dots \quad (*)$

Volume of revolution about x -axis = $\int \pi y^2 dx$

From (*) $y^2 = 4 - (x-1)^2$

$$\begin{aligned}
 \text{Volume} &= \int_0^3 \pi (4 - (x-1)^2) dx \\
 &= \pi \int_0^3 (4 - (x^2 - 2x + 1)) dx \\
 &= \pi \int_0^3 (3 - x^2 + 2x) dx \\
 &= \pi \left[3x - \frac{x^3}{3} + \frac{x^2}{2} \right]_0^3 \\
 &= \pi \left\{ (9 - 9 + 9) - 0 \right\} \\
 &= 9\pi
 \end{aligned}$$



(b) $y = f(x+2)$
 shift 2 to left
 asymptote $y = x - 3$

(c) $x = P(f(x))$
 $\Rightarrow P^{-1}(x) = P^{-1}(P(f(x)))$
 $\Rightarrow f^{-1}(x) = P(x)$ One solution since one point of intersection

$$(12) \int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta d\theta$$

$$\text{limits } x=0 \quad \begin{aligned} \tan\theta &= 0 \\ \theta &= 0 \end{aligned}$$

$$x=1 \quad \begin{aligned} \tan\theta &= 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

Substitute

$$\int_0^{\frac{\pi}{4}} \frac{\sec\theta d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{(\sec^2\theta)^{\frac{3}{2}}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta d\theta$$

$$= [\sin\theta]_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4} - \sin 0$$

$$= \frac{1}{\sqrt{2}}$$

$$(13) F = 15 + e^x (\sin x - \cos x - \sqrt{2})$$

$$\frac{dF}{dx} = e^x (\cos x + \sin x) + e^x (\sin x - \cos x - \sqrt{2})$$

$$= 2e^x \sin x - \sqrt{2}e^x$$

$$= e^x (2 \sin x - \sqrt{2})$$

Check $\frac{dF}{dx} \rightarrow \text{constant}$

For max/min $\frac{dF}{dx} = 0$

$$e^x(2\sin x - \sqrt{2}) = 0$$

$$2\sin x = \sqrt{2}$$

$$\sin x = \frac{1}{\sqrt{2}}$$

For $40 \leq s \leq 120$

$0 \leq x \leq \pi$

so for $\sin x = \frac{1}{\sqrt{2}}$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Note

$$\frac{d^2F}{dx^2} = e^x(2\cos x) + e^x(2\sin x - \sqrt{2})$$

When $x = \frac{\pi}{4}$

$$\frac{d^2F}{dx^2} = e^x \cdot \frac{2}{\sqrt{2}} + e^x \left(\frac{2}{\sqrt{2}} - \sqrt{2} \right)$$

$$= e^x \left(\frac{4}{\sqrt{2}} - \sqrt{2} \right)$$

$$= \sqrt{2}e^x = \sqrt{2}e^{\frac{\pi}{4}}$$

> 0 so $x = \frac{\pi}{4}$ gives a min TP

When $x = \frac{3\pi}{4}$

$$\frac{d^2F}{dx^2} = e^{\frac{3\pi}{4}} \left(-\frac{2}{\sqrt{2}} \right) + e^{\frac{3\pi}{4}} \left(\frac{2}{\sqrt{2}} - \sqrt{2} \right)$$

$$= -\sqrt{2}e^{\frac{3\pi}{4}}$$

< 0 so $x = \frac{3\pi}{4}$ gives a max TP

When $x = \frac{\pi}{4}$

$$\frac{\pi}{4} = \frac{\pi(s-40)}{80}$$

$$20\pi = \pi(s-40)$$

$$s = 60 \text{ km/h}$$

$$\text{and } F = 15 + e^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \sqrt{2} \right) \\ = 15 - \sqrt{2}e^{\frac{\pi}{4}}$$

When $x = \frac{3\pi}{4}$

$$\frac{3\pi}{4} = \frac{\pi}{80}(S-40)$$

$$60 = S-40$$

$$S = 100 \text{ km/h}$$

$$\text{and } F = 15 + e^{\frac{3\pi}{4}} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \sqrt{2} \right) \\ = 15$$

Greatest efficiency is 15 km per l. when $S = 100 \text{ km/h}$.

[Least efficiency is $(15 - \sqrt{2}e^{\frac{\pi}{4}})$ km per l when $S = 60 \text{ km/h}$.]

* Check end points too
(see end)

Q (i4) (a) $1+r+r^2+r^3+\dots$

$$\begin{matrix} a = 1 \\ r = r \end{matrix}$$

$$S_\infty = \frac{a}{1-r} \\ = \frac{1}{1-r}$$

$$\begin{aligned} \frac{1}{2-3r} &= \frac{1}{2} \cdot \frac{1}{1-\frac{3}{2}r} \\ &= \frac{1}{2} \left\{ 1 + \frac{3}{2}r + \left(\frac{3}{2}r\right)^2 + \left(\frac{3}{2}r\right)^3 + \dots \right\} \\ &= \frac{1}{2} + \frac{3}{4}r + \frac{9}{8}r^2 + \frac{27}{16}r^3 + \dots \end{aligned}$$

Converges for $\left|\frac{3}{2}r\right| < 1$

$$\text{i.e. } -1 < \frac{3}{2}r < 1$$

$$-\frac{2}{3} < r < \frac{2}{3}$$

$$\begin{aligned} (b) \quad \frac{1}{3r^2-5r+2} &= \frac{1}{(3r-2)(r-1)} \\ &= \frac{A}{3r-2} + \frac{B}{r-1} \\ &= \frac{A(r-1) + B(3r-2)}{(3r-2)(r-1)} \end{aligned}$$

$$1 = A(r-1) + B(3r-2)$$

$$\text{let } r=1 \quad 1 = B$$

$$\text{let } r = \frac{2}{3} \quad 1 = -\frac{1}{3}A$$

$$A = -3$$

$$\begin{aligned} \text{So } \frac{1}{3r^2-5r+2} &= \frac{-3}{3r-2} + \frac{1}{r-1} \\ &= \frac{1}{r-1} - \frac{3}{3r-2} = \frac{1}{r-1} + \frac{3}{2-3r} = \frac{1}{r-1} + \frac{3}{2-3r} \end{aligned}$$

$$\begin{aligned} \text{So } \frac{1}{3r^2-5r+2} &= -(1+r+r^2+r^3+\dots) + 3\left(\frac{1}{2} + \frac{3}{4}r + \frac{9}{8}r^2 + \dots\right) \\ &= \cancel{\frac{1}{2}}r^2 - \cancel{\frac{5}{4}}r - \cancel{\frac{19}{8}}r^3 + \dots \\ &= \cancel{\frac{5}{2}}r + \cancel{\frac{13}{4}}r^2 + \cancel{\frac{35}{8}}r^3 + \dots = -\frac{1}{2} + \frac{5}{4}r + \frac{19}{8}r^2 + \dots \\ &\text{Converges for } |r| < \frac{2}{3}. \end{aligned}$$

$$15(a) I = \int e^x \cos x \, dx$$

$$\begin{aligned} &= e^x \cdot \sin x - \int \sin x \cdot e^x \, dx \\ &= e^x \sin x - \left\{ e^x \cdot (-\cos x) - \int (-\cos x) e^x \, dx \right\} \\ &= e^x \sin x + e^x \cos x - I + C_1 \end{aligned}$$

$$2I = e^x (\sin x + \cos x) + C_1$$

$$I = \frac{1}{2} e^x (\sin x + \cos x) + C.$$

$$(b) I_n = \int e^x \cos nx \, dx$$

$$\begin{aligned} &= e^x \frac{1}{n} \sin nx - \int \frac{1}{n} \sin nx e^x \, dx \\ &= \frac{1}{n} e^x \sin nx - \frac{1}{n} \left\{ e^x \left(-\frac{1}{n} \cos nx\right) - \int \left(-\frac{1}{n} \cos nx\right) e^x \, dx \right\} \\ &= \frac{1}{n} e^x \sin nx + \frac{1}{n^2} e^x \cos nx - \frac{1}{n^2} I_n + C_1 \end{aligned}$$

$$\left(1 + \frac{1}{n^2}\right) I_n = \frac{1}{n} e^x \sin nx + \frac{1}{n^2} e^x \cos nx + C_1$$

$$\frac{n^2+1}{n^2} I_n = n^2 e^x (n \sin nx + \cos nx) + C_1$$

$$I_n = \frac{1}{(n^2+1)} e^x (n \sin nx + \cos nx) + C$$

$$(c) \int_0^{\frac{\pi}{2}} e^x \cos 8x \, dx$$

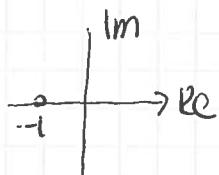
$$= \left[\frac{1}{65} e^x (8 \sin 8x + \cos 8x) \right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{65} e^{\frac{\pi}{2}} (8 \sin 4\pi + \cos 4\pi) \right) - \left(\frac{1}{65} e^0 (8 \sin 0 + \cos 0) \right)$$

$$= \left(\frac{1}{65} e^{\frac{\pi}{2}} (0 + 1) \right) - \frac{1}{65} (1)$$

$$= \frac{1}{65} (e^{\frac{\pi}{2}} - 1)$$

(16)



$$r=1 \quad \theta=\pi$$

$$-i = 1 (\cos \pi + i \sin \pi)$$

$$z^4 + 1 = 0$$

$$z^4 = -1$$

$$z^4 = \cos \pi + i \sin \pi$$

$$z = (\cos(\pi + 2n\pi) + i \sin(\pi + 2n\pi))^{\frac{1}{4}}$$

$$z = \left(\cos \frac{\pi + 2n\pi}{4} + i \sin \frac{\pi + 2n\pi}{4} \right)$$

$$\begin{aligned} n=0 \quad z &= \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} n=1 \quad z &= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \\ &= -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} n=2 \quad z &= \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \end{aligned}$$

$$n=3 \quad z = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

$$(b) \quad z^4 = 1 = \cos 0 + i \sin 0$$

$$z = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{4}} = \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}$$

$$n=0 \quad z = 1$$

$$n=1 \quad z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \\ = i$$

$$\begin{aligned} & \text{or} & z^4 - 1 &= 0 \\ & (z^2 - 1)(z^2 + 1) &= 0 \end{aligned}$$

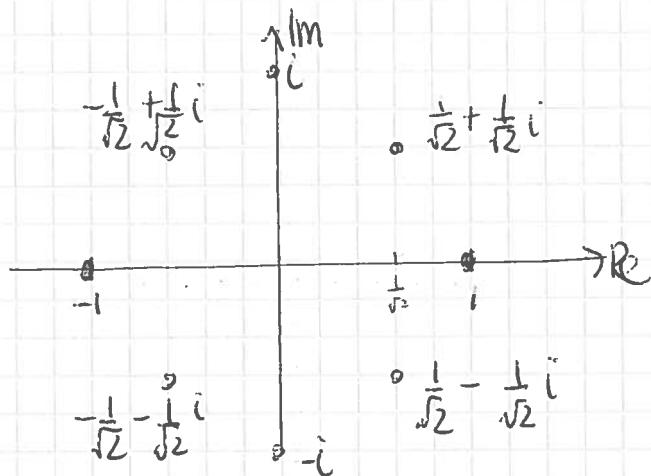
$$n=2 \quad z = -1$$

$$n=3 \quad z = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \\ = -i$$

$$\begin{aligned} z^2 &= 1 & z^2 &= -1 \\ z &= \pm 1 & z &= \pm i \end{aligned}$$

Solutions $\pm 1 \quad \pm i$

(c)



$$(d) \quad z^8 - 1 = 0$$

$$(z^4 - 1)(z^4 + 1) = 0$$

$$z^4 - 1 = 0 \quad \text{or} \quad z^4 + 1 = 0$$

so the solutions of these two equations
are the solutions to $z^8 - 1 = 0$

$$(e) z^6 + z^4 + z^2 + 1 = 0$$

$z^2 = -1$ is a solution i.e. $z^2 + 1$ is a factor

$$z^2 + 1 \mid \overline{\begin{array}{r} z^6 + 1 \\ z^6 + z^4 + z^2 + 1 \\ \hline z^4 + 1 \end{array}}$$

$$z^6 + z^4 + z^2 + 1 = (z^2 + 1)(z^4 + 1) = 0$$

$$\text{so } z^2 + 1 = 0$$

$$z = \pm i$$

$$z^4 + 1 = 0$$

$$z = \pm \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$$

$$\sigma \pm \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right)$$

(B) cont.

$$\text{When } s = 60 \quad x = 0 \quad \rightarrow f = 15 + (-1 - \sqrt{2}) \\ = 14 - \sqrt{2} = 12.6$$

$$\text{When } s = 120 \quad x = \pi \quad f = 15 + e^\pi (1 - \sqrt{2}) \\ = 5.41$$

Conclusion

Greatest efficiency is 15 km per l. when $s = 100$ km/h

least efficiency is $15 + e^\pi (1 - \sqrt{2})$ km per l when $s = 120$ km/h.