

# Advanced Higher Maths 2013.

$$\textcircled{1} \quad \left( 3x - \frac{2}{x^2} \right)^4$$

$$= (3x)^4 + 4(3x)^3 \left( -\frac{2}{x^2} \right) + 6(3x)^2 \left( -\frac{2}{x^2} \right)^2 + 4(3x) \left( -\frac{2}{x^2} \right)^3$$

1			
1	2	1	
1	3	3	1
1	4	6	4

$$+ \left( -\frac{2}{x^2} \right)^4$$

$$= 81x^4 - 216x + \frac{216}{x^2} - \frac{96}{x^5} + \frac{16}{x^8}$$

$$\textcircled{2} \quad f(x) = e^{\cos x} \sin^2 x$$

$$u = e^{\cos x}$$

$$v = \sin^2 x$$

$$p'(x) = e^{\cos x} 2\sin x \cos x + \sin^2 x (-\sin x e^{\cos x})$$

$$u' = e^{\cos x} \cdot -\sin x$$

$$v' = 2\sin x \cos x$$

$$= e^{\cos x} (2\sin x \cos x - \sin^3 x)$$

$$= \sin x \cdot e^{\cos x} (2\cos x - \sin^2 x)$$

$$\textcircled{3} \quad (\text{a}) \quad A^2 = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16-2p & 4p+p \\ -8-2 & -2p+1 \end{pmatrix}$$

$$= \begin{pmatrix} 16-2p & 5p \\ -10 & -2p+1 \end{pmatrix}$$

$$(\text{b}) \quad A^2 \text{ singular} \Rightarrow \det A^2 = 0$$

$$\begin{aligned}
 (16 - 2p)(-2p + 1) + 50p &= 0 \\
 -32p^2 + 16 + 4p^2 - 2p + 50p &= 0 \\
 4p^2 + 16p + 16 &= 0 \\
 p^2 + 4p + 4 &= 0 \\
 (p+2)(p+2) &= 0 \\
 p &= -2.
 \end{aligned}$$

$$(c) B = 3A'$$

$$\begin{aligned}
 \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} &= 3 \begin{pmatrix} 4 & -2 \\ -2p & 1 \end{pmatrix} \\
 \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} &= \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}
 \end{aligned}$$

$$x = 12$$

$$\begin{aligned}
 3p &= 1 \\
 p &= \frac{1}{3}
 \end{aligned}$$

$$(4) v = e^{3t} + 2e^t$$

$$\begin{aligned}
 (a) a &= \frac{dv}{dt} \\
 &= 3e^{3t} + 2e^t
 \end{aligned}$$

$$\begin{aligned}
 (b) \text{ distance} &= \int_0^{\ln 3} (e^{3t} + 2e^t) dt \\
 (\text{area under graph}) &= \left[ \frac{1}{3}e^{3t} + 2e^t \right]_0^{\ln 3}
 \end{aligned}$$

$$\begin{aligned}
 &= \left( \frac{1}{3}e^{3\ln 3} + 2e^{\ln 3} \right) - \left( \frac{1}{3} + 2 \right) \\
 &= 9 + 6 - \frac{1}{3} - 2
 \end{aligned}$$

=

$$= 12 \frac{2}{3}$$

(or find displacement when  $t = \ln 3$   
and displacement when  $t = 0$   
and subtract)

$$\textcircled{5} \quad 1204 = 833x1 + 371$$

$$833 = 371 \times 2 + 91$$

$$371 = 91 \times 4 + 7$$

$$91 = 7 \times 13 + 0$$

$$(1204, 833) = (833, 371)$$

$$= (371, 91)$$

$$= (91, 7)$$

$$= (7, 0)$$

$$\text{gcd} = 7.$$

$$7 = 371 - 91 \times 4.$$

$$7 = 371 - (833 - 371 \times 2) \times 4$$

$$7 = 371 \times 9 - 833 \times 4$$

$$7 = (1204 - 833 \times 1) \times 9 - 833 \times 4$$

$$7 = 1204 \times 9 - 833 \times 13.$$

$$a = 9 \quad b = -13.$$

\textcircled{6}

$$\int \frac{\sec^2 3x}{1 + \tan 3x} dx \quad \frac{d}{dx}(1 + \tan 3x)$$

$$= 3 \sec^2 3x$$

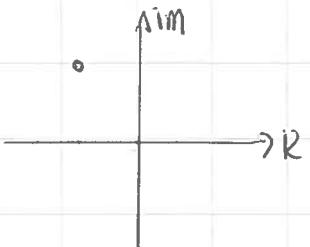
$$= \frac{1}{3} \int \frac{3 \sec^2 3x}{1 + \tan 3x} dx$$

$$= \frac{1}{3} \ln |1 + \tan 3x| + C.$$

$$⑦ \quad z = 1 - \sqrt{3}i$$

$$\bar{z} = 1 + \sqrt{3}i$$

$$\begin{aligned}\bar{z}^2 &= (1 + \sqrt{3}i)^2 \\ &= 1 + 2\sqrt{3}i - 3 \\ &= -2 + 2\sqrt{3}i\end{aligned}$$



$$\begin{aligned}|z|^2 &= \sqrt{4 + (2\sqrt{3})^2} \\ &= \sqrt{4 + 12} \\ &= 4\end{aligned}$$

$$\tan \theta = \frac{2\sqrt{3}}{-2}$$

$$= -\sqrt{3} \quad \text{ra } \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\bar{z}^2 = 4 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$⑧ \quad \int x^2 \cos 3x \, dx$$

$$= x^2 \cdot \frac{1}{3} \sin 3x - \frac{1}{3} \int \sin 3x \cdot 2x \, dx$$

$$= \frac{1}{3} x^2 \sin 3x - \frac{2}{3} \left( x \cdot -\frac{1}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx \right) \text{S}$$

$$= \frac{1}{3} x^2 \sin 3x + \frac{2x}{9} \cos 3x - \frac{2}{27} \sin 3x + C.$$

$$\textcircled{q} \quad \sum_{r=1}^n (4r^3 + 3r^2 + r) = n(n+1)^3$$

$n=1$

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^1 4r^3 + 3r^2 + r \\ &= 4+3+1 \\ &= 8 \end{aligned} \quad \begin{aligned} \text{RHS} &= 1(1+1)^3 \\ &= 2^3 \\ &= 8 \end{aligned}$$

$$\text{so LHS} = \text{RHS}$$

Hence true for  $n=1$

Assume true for  $n=k$ .

$$\text{i.e. } \sum_{r=1}^k (4r^3 + 3r^2 + r) = k(k+1)^3$$

Consider  $n=k+1$

$$\begin{aligned} \sum_{r=1}^{k+1} (4r^3 + 3r^2 + r) &= \sum_{r=1}^k (4r^3 + 3r^2 + r) + 4(k+1)^3 + 3(k+1)^2 + (k+1) \\ &= k(k+1)^3 + 4(k+1)^3 + 3(k+1)^2 + (k+1) \\ &= (k+1)(k(k+1)^2 + 4(k+1)^2 + 3(k+1) + 1) \\ &= (k+1)(k^3 + 2k^2 + k + 4k^2 + 8k + 4 + 3k + 3 + 1) \\ &= (k+1)(k^3 + 6k^2 + 12k + 8) \\ &= (k+1)(k+2)^3 \\ &= (k+1)((k+1)+1)^3 \end{aligned}$$

so true for  $n=k+1$

Result is true for  $n=1$ , and if result is true for  $n=k$  then it is true for  $n=k+1$ , hence by induction the result is true for all  $n \geq 1$

$$\textcircled{10} \quad (a) \quad |z+i| = 1$$

$$\text{Let } z = x+iy$$

$$|x+iy+i| = 1$$

$$|(x+i(y+1))| = 1$$

$$\sqrt{x^2 + (y+1)^2} = 1$$

$$x^2 + (y+1)^2 = 1$$

circle centre  $(0, -1)$  radius 1

$$(b) \quad |z-1| = |z+5|$$

$$\text{Let } z = x+iy$$

$$|x+iy-1| = |x+iy+5|$$

$$|(x-1)+iy| = |(x+5)+iy|$$

$$(x-1)^2 + y^2 = (x+5)^2 + y^2$$

$$x^2 - 2x + 1 + y^2 = x^2 + 10x + 25 + y^2$$

$$-12x = 24$$

$$x = -2$$

vertical line.  $x = -2$ .

$$\textcircled{11} \quad x^2 + 4xy + y^2 + 11 = 0$$

Differentiate

$$2x + 4x \frac{dy}{dx} + 4y + 2y \frac{dy}{dx} = 0$$

$$4x \frac{dy}{dx} + 2y \frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x - 4y}{4x + 2y}$$

$$\frac{dy}{dx} = \frac{-x - 2y}{2x + y}$$

Point  $(-2, 3)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2-6}{-4+3} \\ &= \frac{-4}{-1} \\ &= 4.\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{x-2y}{2x+y} \\ \frac{dy}{dx^2} &= \frac{(2x+y) \cdot (-1 - 2 \frac{dy}{dx}) - (-x-2y) \cdot (2 + \frac{dy}{dx})}{(2x+y)^2}\end{aligned}$$

$$\text{Substitute } x = -2, y = 3 \quad \frac{dy}{dx} = 4$$

$$\begin{aligned}\frac{dy}{dx^2} &= \frac{(-4+3)(-1-8) - (2-6)(2+4)}{(-1)^2} \\ &= \frac{(-1)(-9) - (-4)(6)}{1} \\ &= \frac{9+24}{1} \\ &= 33.\end{aligned}$$

⑫ A. if  $n$  is a multiple of  $q$  then so is  $n^2$ .

True.

$$\text{Let } n = qk$$

$$\begin{aligned}n^2 &= (qk)^2 \\ &= 81k^2\end{aligned}$$

$= 9(qk^2)$  so  $n^2$  is a multiple of 9

Hence if  $n$  is a multiple of  $q$  then so is  $n^2$ .

(B) If  $n^2$  is a multiple of 9 then so is n.

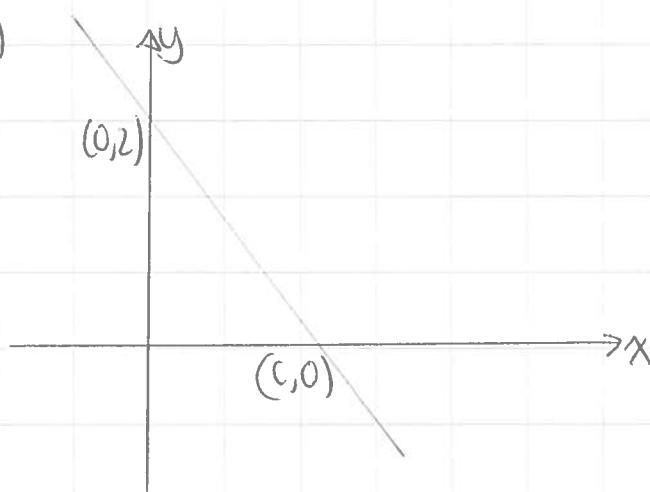
False.

Counterexample  $n^2 = 36$

which is a multiple of 9

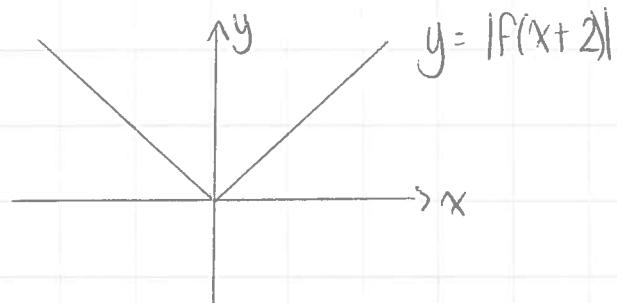
but  $n = 6$  which is not a multiple of 9.

(B). (a)



(b)  $k = -c$ .

(c)  $h = +2$ .



(14)  $\frac{dy}{dx^2} - 6 \frac{dy}{dx} + 9y = 4e^{3x}$

Auxiliary equation

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$m=3.$$

CF  $y = Ae^{3x} + Bxe^{3x}$

PI Try  $y = Cx^2e^{3x}$

$$\frac{dy}{dx} = Cx^2 \cdot 3e^{3x} + 2Cx \cdot e^{3x}$$

$$\frac{d^2y}{dx^2} = 3Cx^2 \cdot 3e^{3x} + 2C \cdot 3e^{3x} + 2Cx \cdot 3e^{3x}$$

$$+ 2Ce^{3x}$$

Substitute.

$$9Cx^2e^{3x} + 18Cxe^{3x} + 2Ce^{3x} - 6(3Cx^2e^{3x} + 2Cx \cdot 3e^{3x})$$

$$+ 9Cx^2e^{3x} = 4e^{3x}$$

$$12Cxe^{3x} + 2Ce^{3x} - 12Cx \cdot 3e^{3x} + 18Cx^2e^{3x} = 4e^{3x}$$

$$2Ce^{3x} = 4e^{3x}$$

$$2C = 4$$

$$C = 2$$

General solution.

$$y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}$$

When  $y=1$ ,  $x=0$

$$1 = A + 0$$

$$A = 1$$

When  $x=0$   $\frac{dy}{dx} = -1$

$$\frac{dy}{dx} = 3Ae^{3x} + Bx \cdot 3e^{3x} + Be^{3x} + 2x^2 \cdot 3e^{3x}$$

$$+ 6xe^{3x}$$

$$-1 = 3 + B$$

$$B = -4$$

Particular solution

$$y = e^{3x} - 4xe^{3x} + 2x^2e^{3x}$$

$$\begin{aligned} \textcircled{15} \quad (a) \quad \vec{AB} &= \underline{b} - \underline{a} & \vec{BC} &= \underline{c} - \underline{b} \\ &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} & &= \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & &= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{AB} \times \vec{BC} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} \\ &= \underline{i}(2-0) - \underline{j}(-1-0) + \underline{k}(0+1) \\ &= 2\underline{i} - 2\underline{j} + \underline{k} \end{aligned}$$

$$\underline{n} \cdot \underline{r} = \underline{n} \cdot \underline{a}$$

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$$2x - 2y + z = 5$$

$$(b) \quad \pi_2$$

$$\underline{n} \cdot \underline{r} = \underline{n} \cdot \underline{a}$$

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$$-y + z = 4$$

$$(c) \quad \underline{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \underline{n}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{n}_1 \cdot \underline{n}_2 = 0 + 2 + 1 \\ = 3.$$

$$|\underline{n}_1| = \sqrt{4+4+1} \\ = 3$$

$$|\underline{n}_2| = \sqrt{1+1} \\ = \sqrt{2}$$

$$\cos\theta = \frac{3}{3 \times \sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \quad (\text{or } 45^\circ)$$

$$(16) \quad \frac{dp}{dt} = p(1000 - p)$$

$$\int \frac{dp}{p(1000-p)} = \int dt$$

Partial Bruchans.

$$\frac{1}{p(1000-p)} = \frac{A}{p} + \frac{B}{1000-p}$$

$$1 = A(1000-p) + B(p)$$

$$\text{het } p = 0 \quad A = \frac{1}{1000}$$

$$\text{het } p = 1000 \quad B = \frac{1}{1000}$$

$$\int_{1000}^P \left( \frac{1}{P} + \frac{1}{1000-P} \right) dP = \int dt$$

$$\int \left( \frac{1}{P} + \frac{1}{1000-P} \right) dP = \int 1000 dt$$

$$\ln|P| - \ln|1000-P| = 1000t + C.$$

$$\ln \frac{P}{1000-P} = 1000t + C. \quad \text{as required}$$

$$\frac{P}{1000-P} = e^{1000t+C}$$

$$\frac{P}{1000-P} = e^C e^{1000t}$$

$$\frac{P}{1000-P} = k e^{1000t}$$

$$P = (1000-P) k e^{1000t}$$

$$P + P k e^{1000t} = 1000 k e^{1000t}$$

$$P = \frac{1000 k e^{1000t}}{1 + k e^{1000t}}$$

Divide by  $e^{1000t}$

$$P(t) = \frac{1000k}{e^{-1000t} + k} \quad \text{as required.}$$

$$P(0) = 200$$

$$200 = \frac{1000k}{1+k}$$

$$200(1+k) = 1000k$$

$$200 = 800k$$

$$k = \frac{1}{4}.$$

$$P(t) = \frac{250}{e^{-1000t} + \frac{1}{4}}$$

$$900 = \frac{250}{e^{-1000t} + \frac{1}{4}}$$

$$e^{-1000t} + \frac{1}{4} = \frac{250}{900}$$

$$e^{-1000t} = \frac{250}{900} - \frac{1}{4}$$

$$e^{-1000t} = \frac{1}{36}$$

$$e^{1000t} = 36$$

$$1000t = \ln 36$$

$$t = \frac{\ln 36}{1000}$$

$$(17) \quad 1 + x + x^2 + x^3 + \dots \quad S_\infty = \frac{1}{1-x}$$

$$1 - x + x^2 - x^3 + \dots \quad S_\infty = \frac{1}{1+x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Integrate  $-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Integrate

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right) \\ &= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)\end{aligned}$$

$$\frac{1+x}{1-x} = 2$$

$$1+x = 2-2x$$

$$3x = 1$$

$$x = \frac{1}{3}$$

let  $x = \frac{1}{3}$  in expansion

$$\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = 2\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \dots\right)$$

$$\begin{aligned}\ln 2 &= 2\left(0.33333 + 0.01234 + 0.000823 + 0.000065\dots\right) \\ &= 0.693 \quad (3 \text{ d.p.})\end{aligned}$$