X100/13/01

NATIONAL QUALIFICATIONS 2013

WEDNESDAY, 22 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1 Calculators may be used in this paper.

2 Candidates should answer **all** questions.

3 **Full credit will be given only where the solution contains appropriate working.**

Marks

Answer all the questions

1. Write down the binomial expansion of $\left(3x - \frac{2}{x^2}\right)^4$ and simplify your answer. **4**

2. Differentiate $f(x) = e^{\cos x}\sin^2 x$. **3**

3. Matrices A and B are defined by $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$.

   (*a*) Find A^2. **1**

   (*b*) Find the value of p for which A^2 is singular. **2**

   (*c*) Find the values of p and x if $B = 3A'$. **2**

4. The velocity, v, of a particle P at time t is given by

   $$v = e^{3t} + 2e^t.$$

   (*a*) Find the acceleration of P at time t. **2**

   (*b*) Find the distance covered by P between $t = 0$ and $t = \ln 3$. **3**

5. Use the Euclidean algorithm to obtain the greatest common divisor of 1204 and 833, expressing it in the form $1204a + 833b$, where a and b are integers. **4**

6. Integrate $\dfrac{\sec^2 3x}{1 + \tan 3x}$ with respect to x. **4**

7. Given that $z = 1 - \sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form. **4**

8. Use integration by parts to obtain $\int x^2 \cos 3x \, dx$. **5**

9. Prove by induction that, for all positive integers n,

   $$\sum_{r=1}^{n} \left(4r^3 + 3r^2 + r\right) = n(n+1)^3 \quad .$$ **6**

Marks

10. Describe the loci in the complex plane given by:

(*a*) $|z + i| = 1$; **2**

(*b*) $|z - 1| = |z + 5|$. **3**

11. A curve has equation

$$x^2 + 4xy + y^2 + 11 = 0.$$

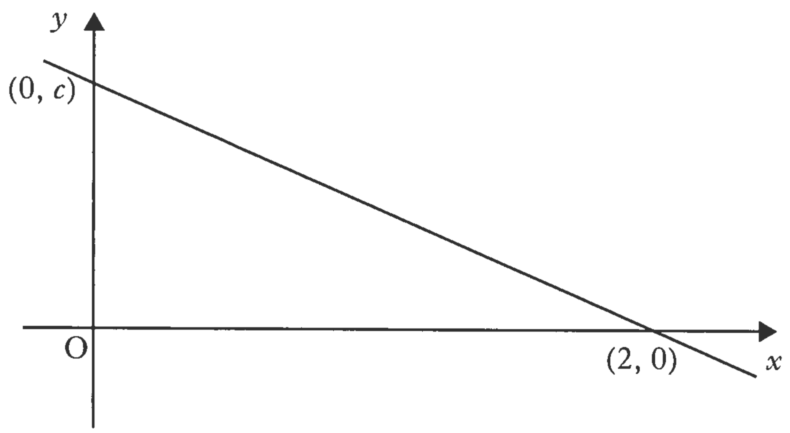
Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$. **6**

12. Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9 then so is n^2.

B If n^2 is a multiple of 9 then so is n. **4**

13. Part of the straight line graph of a function $f(x)$ is shown.



(*a*) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes. **2**

(*b*) State the value of k for which $f(x) + k$ is an odd function. **1**

(*c*) Find the value of h for which $|f(x + h)|$ is an even function. **2**

[Turn over for Questions 14 to 17 on *Page four*

14. Solve the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}, \text{ given that } y = 1 \text{ and } \frac{dy}{dx} = -1 \text{ when } x = 0.$$ **11**

15. (*a*) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$. **4**

(*b*) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.

Find an equation of the plane π_2. **2**

(*c*) Determine the acute angle between planes π_1 and π_2. **3**

16. In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that

$$\ln\frac{P}{1000 - P} = 1000t + C \quad \text{for some constant } C.$$ **4**

Hence show that

$$P(t) = \frac{1000K}{K + e^{-1000t}} \quad \text{for some constant } K.$$ **3**

Given that $P(0) = 200$, determine at what time t, $P(t) = 900$. **3**

17. Write down the sums to infinity of the geometric series

$1 + x + x^2 + x^3 + \ldots\ldots\ldots$

and

$1 - x + x^2 - x^3 + \ldots\ldots\ldots$

valid for $|x| < 1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\right).$$ **7**

Show how this series can be used to evaluate $\ln 2$.

Hence determine the value of $\ln 2$ correct to 3 decimal places. **3**

[*END OF QUESTION PAPER*]