

# Advanced Higher 2012

① (a)  $f(x) = \frac{3x+1}{x^2+1}$

$$\begin{aligned}f'(x) &= (x^2+1).3 - (3x+1).2x \\&\quad (x^2+1)^2 \\&= \frac{3x^2+3 - 6x^2 - 2x}{(x^2+1)^2} \\&= \frac{3 - 2x - 3x^2}{(x^2+1)^2}\end{aligned}$$

(b)  $g(x) = \cos^2 x e^{\tan x}$

$$\begin{aligned}g'(x) &= \cos^2 x \cdot e^{\tan x} \cdot \sec^2 x + e^{\tan x} \cdot 2\cos x \cdot -\sin x \\&= \cos^2 x \cdot e^{\tan x} \cdot 1 + -2\sin x \cos x e^{\tan x} \\&= e^{\tan x} - \sin 2x e^{\tan x} \\&= e^{\tan x} (1 - \sin 2x)\end{aligned}$$

②  $u_1 = 2048 \qquad u_4 = 256$   
 $a = 2048 \qquad ar^3 = 256$

$$2048 a^3 = 256$$

$$a^3 = \frac{1}{8}$$

$$a = \frac{1}{2}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$4088 = 2048 \frac{\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

$$\frac{2044}{2048} = 1 - \left(\frac{1}{2}\right)^n$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{512}$$

$$2^n = 512$$

$$n = \log_2 512$$

$$\underline{n = 9}$$

③  $(-1+2i)$  is a root so  $(-1-2i)$  is also a root

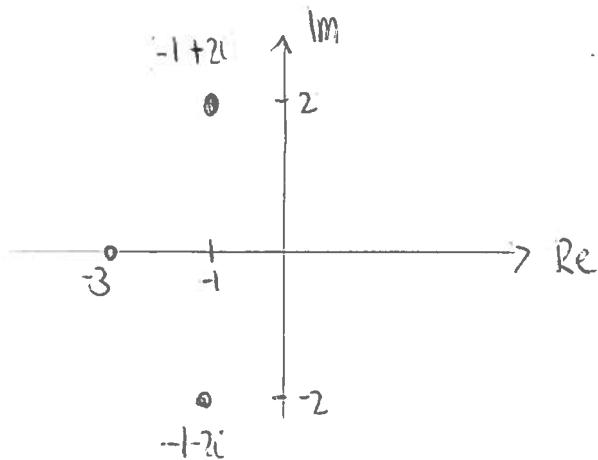
$$\begin{aligned} & (z - (-1+2i))(z - (-1-2i)) \\ &= z^2 - z(-1+2i) - z(-1-2i) + (-1+2i)(-1-2i) \\ &= z^2 + z - 2iz + z + 2iz + 1 + 4 \\ &= z^2 + 2z + 5 \end{aligned}$$

$$\begin{array}{r|rr} & z+3 & \\ \hline z^2 + 2z + 5 & | z^3 + 5z^2 + 11z + 15 \\ & z^3 + 2z^2 + 5z & \\ \hline & 3z^2 + 6z + 15 & \\ & 3z^2 + 6z + 15 & \end{array}$$

$$\text{So } z^3 + 5z^2 + 11z + 15 = 0$$

$$\Rightarrow (z+3)(z - (-1+2i))(z - (-1-2i)) = 0$$

$$z = -3, -1+2i, -1-2i$$



$$④ \left( 2x - \frac{1}{x^2} \right)^9$$

General Term

$$\begin{aligned}
 & {}^9C_r (2x)^{9-r} \left(-\frac{1}{x^2}\right)^r \\
 &= {}^9C_r 2^{9-r} x^{9-r} (-x^2)^r \\
 &= {}^9C_r 2^{9-r} x^{9-r} (-1)^r x^{-2r} \\
 &= {}^9C_r 2^{9-r} (-1)^r x^{9-3r}
 \end{aligned}$$

term independent of  $x$

$$\begin{aligned}
 9-3r &= 0 \\
 r &= 3.
 \end{aligned}$$

When  $r=3$  
$${}^9C_3 2^6 (-1)^3 \cdot x^0$$

$$\begin{aligned}
 &= 84 \times 64 \times -1 \\
 &= \underline{\underline{-5376}}.
 \end{aligned}$$

$$⑤ P(-2, 1, -1) \quad Q(1, 2, 3) \quad R(3, 0, 1)$$

$$\begin{aligned}\vec{PQ} &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{QR} &= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}\end{aligned}$$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 2 & -2 & -2 \end{vmatrix}$$

$$\begin{aligned}&= i(-2+8) - j(-6-8) + k(-6-2) \\&= 6i + 14j - 8k \\&= 2(3i + 7j - 4k)\end{aligned}$$

$$\underline{n} = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\underline{n} \cdot \underline{\Gamma} = \underline{n} \cdot \underline{q}$$

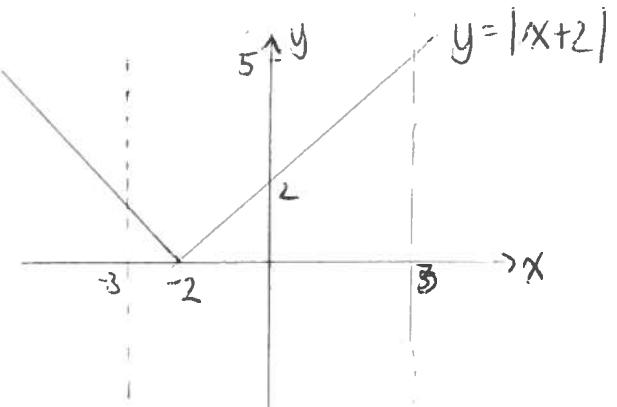
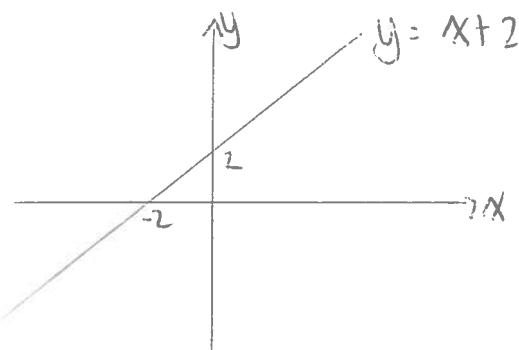
$$\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$\underline{3x + 7y - 4z = 5}$$

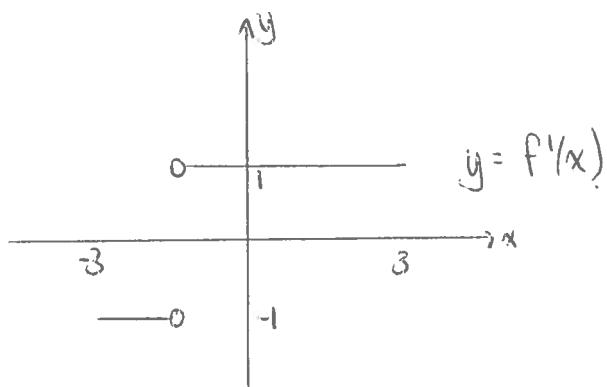
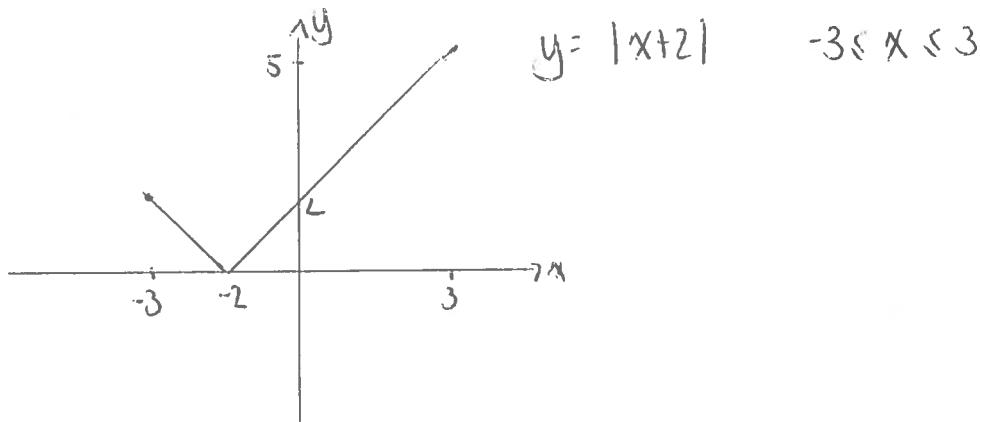
$$⑥ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned}
 (1+e^x)^2 &= 1 + 2e^x + e^{2x} \\
 &= 1 + 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \\
 &\quad + 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \\
 &= 1 + 2 + 2x + x^2 + \frac{x^3}{3} + 1 + 2x + 2x^2 + \frac{4x^3}{3} \\
 &\quad + \dots \\
 &= 4 + 4x + 3x^2 + \frac{5x^3}{3} + \dots
 \end{aligned}$$

$$⑦ f(x) = |x+2|$$



(see over)



$$⑧ \int_0^2 \sqrt{16-x^2} dx$$

let  $x = 4\sin\theta$   
 $dx = +4\cos\theta d\theta$

limits  $x=0 \quad 0 = 4\sin\theta$   
 $\theta = 0$

$x=2 \quad 2 = 4\sin\theta$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

Substitute

$$\int_0^{\frac{\pi}{6}} \sqrt{16-(4\sin\theta)^2} \cdot (+4\cos\theta) d\theta$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{6}} \sqrt{16(1-\sin^2\theta)} \cdot + 4\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{6}} 4\cos\theta (+4\cos\theta) d\theta \\
&= 16 \int_0^{\frac{\pi}{6}} \cos^2\theta d\theta \\
&= 16 \int_0^{\frac{\pi}{6}} \frac{1}{2} (1 + \cos 2\theta) d\theta \\
&= 8 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}} \\
&= 8 \left( \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} \right) - 8(0+0) \\
&= \frac{4\pi}{3} + 4 \cdot \frac{\sqrt{3}}{2} \\
&= \underline{\underline{\frac{4\pi}{3} + 2\sqrt{3}}}.
\end{aligned}$$

⑨  $A + A^{-1} = I$

$$\begin{aligned}
\Rightarrow A(A + A^{-1}) &= A \\
A^2 + I &= A \quad \Rightarrow \quad A^2 = A - I \quad (*) \\
A(A^2 + I) &= A^2 \\
A^3 + A &= A^2 \\
A^3 + A &= A - I \quad (\text{from } (*)) \\
A^3 &= -I \quad R = -1
\end{aligned}$$

$$\begin{array}{rcl}
 \textcircled{10} & 1234 \div 7 & = 176 \quad \text{remainder } 2 \\
 & 176 \div 7 & = 25 \quad \text{remainder } 1 \\
 & 25 \div 7 & = 3 \quad \text{remainder } 4 \\
 & 3 \div 7 & = 0 \quad \text{remainder } 3
 \end{array}$$

$$1234_{10} = 3412_7.$$

$$\textcircled{11} \quad (a) \quad \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$(b) \quad \int \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x \cdot -(1-x^2)^{\frac{1}{2}} + \int (1-x^2)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= -(1-x^2)^{\frac{1}{2}} \sin^{-1} x + \int 1 dx$$

$$= x - (1-x^2)^{\frac{1}{2}} \sin^{-1} x + C.$$

$$\begin{aligned}
 & \int \frac{x}{(1-x^2)^{\frac{1}{2}}} dx \\
 &= -\frac{1}{2} \int -2x (1-x^2)^{-\frac{1}{2}} dx \\
 & \quad \uparrow \quad \uparrow \\
 & \quad \text{derivative} \quad \text{function} \\
 &= -\frac{1}{2} \left( \frac{1-x^2}{\frac{1}{2}} \right)^{\frac{1}{2}} + C \\
 &= -(1-x^2)^{\frac{1}{2}} + C.
 \end{aligned}$$

$$\textcircled{12} \quad V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} + \pi h \cdot 2r \frac{dr}{dt}$$

$$= \pi \times 0.6^2 \cdot 0.01 + \pi \times 2 \times 1.2 \times (-0.02)$$

$$= \pi (0.0036 - 0.048)$$

$$= \underline{-0.0444\pi}.$$

$$(3) \quad x = 2t + \frac{1}{2}t^2 \quad y = \frac{1}{3}t^3 - 3t$$

$$\frac{dx}{dt} = 2 + t \quad \frac{dy}{dt} = 3 \cdot \frac{1}{3}t^2 - 3 \\ = t^2 - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \\ = (t^2 - 3) \cdot \frac{1}{2+t} \\ = \frac{t^2 - 3}{t+2}.$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) \\ &= \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= \frac{(t+2)2t - (t^2 - 3)}{(t+2)^2} \cdot \frac{1}{(2+t)} \\ &= \frac{2t^2 + 4t - t^2 + 3}{(t+2)^3} \\ &= \frac{t^2 + 4t + 3}{(t+2)^3}\end{aligned}$$

For stationary points  $\frac{dy}{dx} = 0$

$$t^2 - 3 = 0$$

$$t = \pm \sqrt{3}$$

When  $t = \sqrt{3}$

$$\frac{d^2y}{dx^2} = \frac{3 + 4\sqrt{3} + 3}{(\sqrt{3} + 2)^3} > 0$$

so  $t = \sqrt{3}$  gives a minimum TP

When  $t = -\sqrt{3}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{3 - 4\sqrt{3} + 3}{(-\sqrt{3} + 2)^3} \\ &= \frac{-0.928}{0.092} \\ &< 0\end{aligned}$$

so  $t = -\sqrt{3}$  gives a maximum TP.

Points of inflection

$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

$$t^2 + 4t + 3 = 0$$

$$(t+3)(t+1) = 0$$

$$t = -3 \text{ and } t = -1$$

i.e. exactly two points of inflection

when  $t = -3$  and when  $t = -1$

(ii) (i)

$$\left( \begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 2 & -2 & 4 & -1 \\ -1 & 1 & \lambda & 2 \end{array} \right)$$

R1 - 2R2

$$\left( \begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & -3 \\ -1 & 1 & \lambda & 2 \end{array} \right)$$

R1 + 4R3

$$\left( \begin{array}{ccc|c} 0 & 4 & 6+4\lambda & 9 \end{array} \right)$$

$$R3 - R2 \left( \begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & 0 & 8+42 & 6 \end{array} \right) \quad * \text{ Missed bit. } \rightarrow \text{ see end}$$

(b) When  $\lambda = -2$  last row  $0 \ 0 \ 0 \ 6$ .  
 ie inconsistent row  
 $\Rightarrow$  no solution.

(c) When  $\lambda = -2.1$

$$\begin{aligned} -0.4z &= 6 \\ z &= -15 \\ \Rightarrow 4y + 30 &= 3 \\ 4y &= -27 \\ y &= -6.75 \\ 4x - 90 &= 1 \\ 4x &= 91 \\ x &= 22.75. \end{aligned}$$

Equations are ill-conditioned. A small change in the value of  $\lambda$  leads to a big change in the solution.

$$\textcircled{15} \quad \frac{1}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2}{(x-1)(x+2)^2} + \frac{B(x-1)(x+2)}{(x-1)(x+2)^2} + \frac{C(x-1)}{(x-1)(x+2)}$$

$$\text{So } 1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$$\text{let } x=1 \quad 1 = A(3^2)$$

$$A = \frac{1}{9}$$

$$\text{let } x=-2 \quad 1 = -3C$$

$$C = -\frac{1}{3}$$

$$\text{let } x=0 \quad 1 = \frac{4}{9} - 2B + \frac{1}{3}$$

$$\frac{2}{9} = -2B$$

$$B = -\frac{1}{9}$$

$$\text{So } \frac{1}{(x-1)(x+2)^2} = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$$

$$(b) \quad (x-1) \frac{dy}{dx} - y = \frac{x-1}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x-1)} \quad y = \frac{1}{(x+2)^2}$$

Integrating factor

$$\begin{aligned} & e^{\int -\frac{1}{x-1} dx} \\ &= e^{-\ln(x-1)} \\ &= e^{\ln \frac{1}{x-1}} \\ &= \frac{1}{x-1} \end{aligned}$$

So

$$\frac{d}{dx} \left( \frac{1}{(x-1)} y \right) = \frac{1}{(x-1)(x+2)^2}$$

$$\begin{aligned} \frac{1}{(x-1)} y &= \int \frac{1}{(x-1)(x+2)^2} dx \\ &= \int \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2} dx \\ \frac{1}{(x-1)} y &= \frac{1}{9} \ln|x-1| - \frac{1}{9} \ln|x+2| + \frac{1}{3} (x+2)^{-1} + C \\ y &= \left( \frac{x-1}{9} \right) \ln \frac{|x-1|}{|x+2|} + \frac{(x-1)}{3(x+2)} + C(x-1) \end{aligned}$$

(16)

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

n=1

$$\begin{aligned} \text{LHS} &= (\cos \theta + i \sin \theta)^1 \\ &= \cos \theta + i \sin \theta \end{aligned}$$

RHS =  $\cos \theta + i \sin \theta$

so true for n=1

Assume result is true for  $n=k$

$$\text{i.e. } (\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$$

Consider  $n=k+1$

$$\begin{aligned}
 (\cos \theta + i \sin \theta)^{k+1} &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\
 &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\
 &= \cos k\theta \cos \theta - \sin k\theta \sin \theta + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta) \\
 &= \cos(k\theta + \theta) + i \sin(k\theta + \theta) \\
 &= \cos((k+1)\theta) + i \sin((k+1)\theta)
 \end{aligned}$$

so true for  $n=k+1$

Result is true for  $n=1$  and if result is true for  $n=k$   
 then it is also true for  $n=k+1$ , hence by induction  
 the result is true for all  $n \geq 1$ .

$$\begin{aligned}
 (b) \quad & \left( \cos \frac{\pi}{18} + i \sin \frac{\pi}{18} \right)^{11} \\
 & \left( \cos \frac{\pi}{36} + i \sin \frac{\pi}{36} \right)^4 \\
 &= \frac{\cos \frac{11\pi}{18} + i \sin \frac{11\pi}{18}}{\left( \cos \frac{\pi}{9} + i \sin \frac{\pi}{9} \right)} \times \left( \cos \frac{\pi}{9} - i \sin \frac{\pi}{9} \right) \\
 &= \frac{\cos \frac{11\pi}{18} \cos \frac{\pi}{9} + \sin \frac{11\pi}{18} \sin \frac{\pi}{9}}{\left( \cos^2 \frac{\pi}{9} - \sin^2 \frac{\pi}{9} \right)} + \frac{i \sin \frac{11\pi}{18} \cos \frac{\pi}{9} - i \cos \frac{11\pi}{18} \sin \frac{\pi}{9}}{\left( \cos^2 \frac{\pi}{9} - \sin^2 \frac{\pi}{9} \right)} \\
 &= \cos \left( \frac{11\pi}{18} - \frac{\pi}{9} \right) + i \sin \left( \frac{11\pi}{18} - \frac{\pi}{9} \right)
 \end{aligned}$$

Since  $\cos \left( \frac{11\pi}{18} - \frac{\pi}{9} \right) = \cos \frac{9\pi}{18} = \cos \frac{\pi}{2} = 0$  the real part is zero

\*  
Hinweis

$$(8+4\lambda)z = 6$$

$$z = \frac{6}{8+4\lambda}$$

$$= \frac{3}{4+2\lambda}$$

$$4y - 2z = 3$$

$$4y - \frac{6}{4+2\lambda} = 3$$

$$\begin{aligned} 4y &= 3 + \frac{6}{4+2\lambda} & 3(4+2\lambda) &+ \frac{6}{4+2\lambda} \\ &= \frac{12+6\lambda+6}{4+2\lambda} & (4+2\lambda) & \end{aligned}$$

$$4y = \frac{18+6\lambda}{4+2\lambda}$$

$$4y = \frac{9+3\lambda}{2+\lambda}$$

$$y = \frac{9+3\lambda}{8+4\lambda}$$

$$4x + 6z = 1$$

$$4x + \frac{18}{4+2\lambda} = 1$$

$$4x = 1 - \frac{18}{4+2\lambda}$$

$$4x = \frac{4+2\lambda-18}{4+2\lambda} = \frac{2\lambda-14}{4+2\lambda} = \frac{\lambda-7}{2+\lambda}$$

$$x = \frac{\lambda-7}{8+4\lambda}$$