

X100/701

NATIONAL
QUALIFICATIONS
2011

WEDNESDAY, 18 MAY
1.00 PM – 4.00 PM

**MATHEMATICS
ADVANCED HIGHER**

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions.

1. Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain

$$\int \frac{13-x}{x^2+4x-5} dx.$$

5

2. Use the binomial theorem to expand $\left(\frac{1}{2}x - 3\right)^4$ and simplify your answer.

3

3. (a) Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation

$$y + e^y = x^2.$$

3

- (b) Given $f(x) = \sin x \cos 2x$, use logarithmic differentiation to obtain $f'(x)$.
 Given $f(x) = \sin x \cos^3 x$ obtain $f'(x)$
 Express $f'(x)$ in the form $\frac{g(x)f(x)}{\sin x \cos x}$.

3

4. (a) For what value of λ is $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$ singular?

3

- (b) For $A = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$, obtain values of α and β such that

$$A' = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

3

5. Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$, and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$.

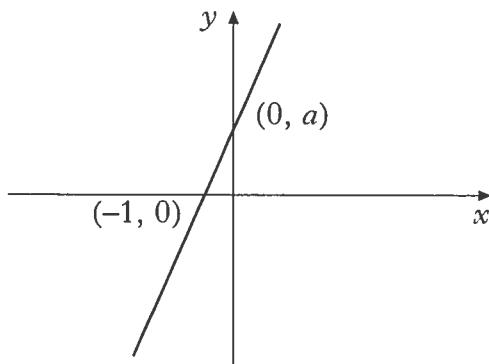
4

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)(1+x^2)}$.

2

Marks

6.



The diagram shows part of the graph of a function $f(x)$. Sketch the graph of $|f^{-1}(x)|$ showing the points of intersection with the axes.

4

7. A curve is defined by the equation $y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$ for $x < 1$.

Calculate the gradient of the curve when $x = 0$.

4

8. Write down an expression for $\sum_{r=1}^n r^3 - \left(\sum_{r=1}^n r \right)^2$

1

and an expression for

$$\sum_{r=1}^n r^3 + \left(\sum_{r=1}^n r \right)^2.$$

3

9. Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

expressing your answer in the form $y = f(x)$.

5

[Turn over

Marks

10. Identify the locus in the complex plane given by

$$|z - 1| = 3.$$

Show in a diagram the region given by $|z - 1| \leq 3$.

5

11. (a) Obtain the exact value of $\int_0^{\pi/4} (\sec x - x)(\sec x + x) dx$.

3

(b) Find $\int \frac{x}{\sqrt{1-49x^4}} dx$.

4

12. Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all integers $n \geq 2$.

5

13. The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$.

Obtain the value of a and the common difference.

5

Obtain the smallest value of n for which the sum of the first n terms is greater than 1000.

4

14. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12.$$

7

Find the particular solution for which $y = -\frac{3}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$ when $x = 0$.

3

Marks

15. The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \text{ and } \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find:

- (a) the value of k for which L_1 and L_2 intersect and the point of intersection; 6
 (b) the acute angle between L_1 and L_2 . 4

16. Define $I_n = \int_0^1 \frac{1}{(1+x^2)^n} dx$ for $n \geq 1$.

- (a) Use integration by parts to show that

$$I_n = \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx. \quad 3$$

- (b) Find the values of A and B for which

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

and hence show that

$$I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n} \right) I_n. \quad 5$$

- (c) Hence obtain the exact value of $\int_0^1 \frac{1}{(1+x^2)^3} dx$. 3

[END OF QUESTION PAPER]

①

AH Maths 2011

$$\begin{aligned}
 ① \quad \frac{B-x}{x^2+4x-5} &= \frac{B-x}{(x+5)(x-1)} \\
 &= \frac{A}{x+5} + \frac{B}{x-1} \\
 &= \frac{A(x-1) + B(x+5)}{(x+5)(x-1)}
 \end{aligned}$$

$$B-x = A(x-1) + B(x+5)$$

$$\text{let } x=1 \quad 12 = 6B$$

$$B = 2$$

$$\text{let } x=-5 \quad 18 = -6A$$

$$A = -3.$$

$$\text{So } \frac{B-x}{x^2+4x-5} = \frac{-3}{x+5} + \frac{2}{x-1}$$

$$\int \frac{B-x}{x^2+4x-5} dx = \int \left(\frac{-3}{x+5} + \frac{2}{x-1} \right) dx$$

$$= -3 \ln|x+5| + 2 \ln|x-1| + C$$

$$= \ln \left| \frac{(x-1)^2}{(x+5)^3} \right| + C.$$

(2)

$$\textcircled{2} \quad \left(\frac{1}{2}x - 3 \right)^4$$

$$\begin{array}{r} 1 \\ 1 \\ 121 \\ 1331 \\ 14641 \end{array}$$

$$= \left(\frac{1}{2}x \right)^4 + 4 \left(\frac{1}{2}x \right)^3 (-3) + 6 \left(\frac{1}{2}x \right)^2 (-3)^2 + 4 \left(\frac{1}{2}x \right) (-3)^3 + (-3)^4$$

$$= \frac{x^4}{16} - 6x^3 + 27x^2 - 54x + 81$$

$$\textcircled{3} \quad (\text{a}) \quad y + e^y = x^2$$

Differentiate

$$\frac{dy}{dx} + e^y \cdot \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} (1 + e^y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{1+e^y}$$

$$\text{(b)} \quad f(x) = \sin x \cos^3 x$$

$$\begin{aligned} f'(x) &= \sin x \cdot 3\cos^2 x \cdot (\sin x) + \cos^3 x \cdot \cos x \\ &= -3\sin^2 x \cos^2 x + \cos^4 x. \end{aligned}$$

$$\textcircled{4} \quad (\text{a}) \quad \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$$

$$\det = 1 \begin{vmatrix} 0 & 2 \\ \lambda & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ -1 & 6 \end{vmatrix} - 1 \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix}$$

(3)

$$\begin{aligned}
 &= (0 - 2\lambda) - 2(18 + 2) - (3\lambda - 0) \\
 &= -2\lambda - 40 - 3\lambda \\
 &= -5\lambda - 40
 \end{aligned}$$

Singular $\Rightarrow \det = 0$

$$\begin{aligned}
 -5\lambda - 40 &= 0 \\
 \lambda &= -8.
 \end{aligned}$$

(b)

$$A = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$$

$$A' = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

So $3\alpha + 2\beta \approx 2\alpha - \beta$ $3\alpha + 2\beta = -5 \dots (1)$
 $2\alpha - \beta = \dots (2)$

$$(2) \times 2 \quad 4\alpha - 2\beta = -2$$

$$(1) \quad 3\alpha + 2\beta = -5$$

$$\text{add} \quad 7\alpha = -7$$

$$\underline{\alpha = -1} \quad \Rightarrow \quad \underline{\beta = -1}.$$

(4)

$$⑤ \quad F(x) = \sqrt{1+x}$$

$$F(0) = 1$$

$$= (1+x)^{\frac{1}{2}}$$

$$F'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$F'(0) = \frac{1}{2}$$

$$F''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$F''(0) = -\frac{1}{4}$$

$$F'''(x) = +\frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$F'''(0) = \frac{3}{8}$$

$$P(x) = f(0) + x F'(0) + \frac{x^2}{2!} F''(0) + \frac{x^3}{3!} F'''(0) + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \dots$$

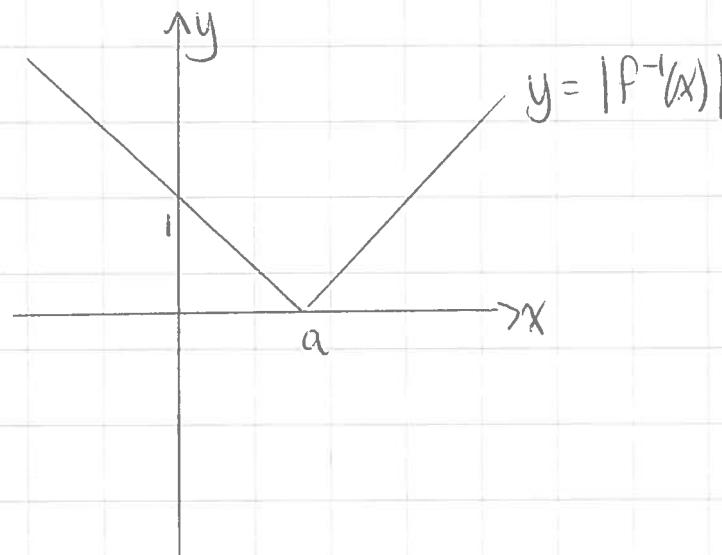
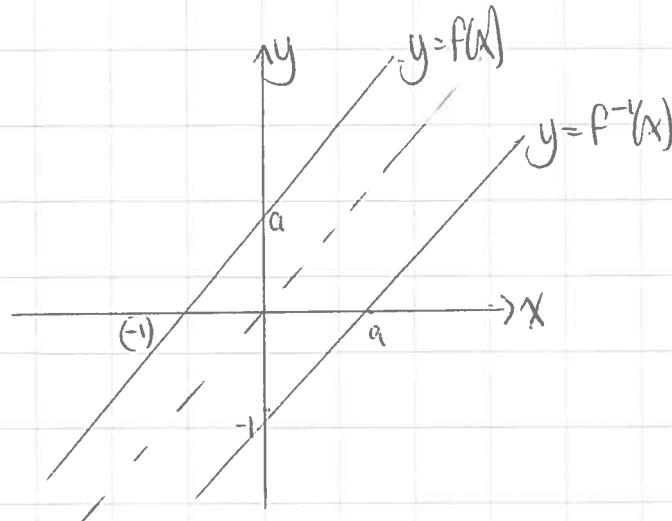
$$\begin{aligned} \sqrt{(1+x)(1+x^2)} &= \sqrt{1+x} \sqrt{1+x^2} \\ &= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots\right) \left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots\right) \\ &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{16}x^5 - \frac{1}{8}x^2 \end{aligned}$$

$$-\frac{1}{16}x^4 + \frac{1}{16}x^3 + \dots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$$

(5)

⑥



⑦

$$y = \frac{e^{\sin x} (2+x)^3}{\sqrt{1-x}}$$

$$\frac{dy}{dx} = (1-x)^{\frac{1}{2}} \cdot \left[e^{\sin x} \cdot 3(2+x)^2 + \cos x e^{\sin x} (2+x)^3 \right] - e^{\sin x} (2+x)^3 \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$\begin{aligned} &= \frac{2(1-x) \left(3e^{\sin x} (2+x)^2 + \cos x e^{\sin x} (2+x)^3 \right) + e^{\sin x} (2+x)^3}{2(1-x)^{\frac{3}{2}}} \\ &= \frac{e^{\sin x} (2+x)^2 [2(1-x) \cdot 3 + 2(1-x)(2+x)(\cos x + 2+x)]}{2(1-x)^{\frac{3}{2}}} \end{aligned}$$

(6)

$$= e^{\sin x} (2+x)^2 (6-6x+2+x + 2(1-x)(2+x)\cos x) \\ = \frac{e^{\sin x} (2+x)^2 (8-5x+2(1-x)(2+x)\cos x)}{2(1-x)^{\frac{3}{2}}}$$

When $x=0$

$$\frac{dy}{dx} = \frac{e^0 2^2 (8+2(1)(2)-1)}{2} \\ = 2(8+4) \\ = 24.$$

⑧ $\sum_{r=1}^n r^3 - \left(\sum_{r=1}^n r\right)^2$

$$= \frac{n^2(n+1)^2}{4} - \left(\frac{1}{2}n(n+1)\right)^2$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)^2}{4}$$

$$= 0$$

$$\sum_{r=1}^n r^3 + \left(\sum_{r=1}^n r\right)^2$$

$$= \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2(n+1)^2}{2}.$$

(7)

$$\textcircled{9} \quad \frac{dy}{dx} = 3(1+y) \sqrt{1+x}$$

$$\int \frac{dy}{1+y} = \int 3(1+x)^{\frac{1}{2}} dx$$

$$\ln|1+y| = \frac{2}{3} \cdot 3 \cdot (1+x)^{\frac{3}{2}} + C.$$

$$\ln|1+y| = 2(1+x)^{\frac{3}{2}} + \ln k.$$

$$\ln\left(\frac{1+y}{k}\right) = 2(1+x)^{\frac{3}{2}}$$

$$\frac{1+y}{k} = e^{2(1+x)^{\frac{3}{2}}}$$

$$1+y = ke^{2(1+x)^{\frac{3}{2}}}$$

$$y = ke^{2(1+x)^{\frac{3}{2}}} - 1.$$

$$\textcircled{10} \quad |z-1| = 3.$$

$$\text{let } z = x+iy$$

$$|x+iy-1| = 3$$

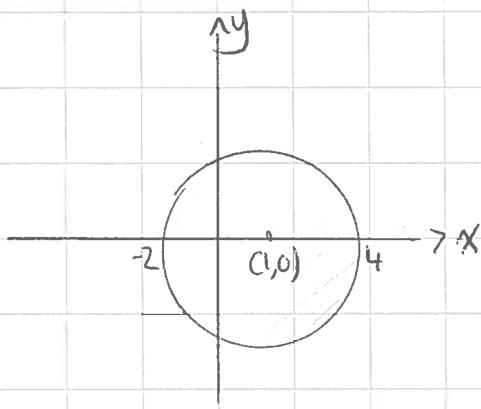
$$|(x-1)+iy| = 3$$

$$\sqrt{(x-1)^2 + y^2} = 3$$

$$(x-1)^2 + y^2 = 9.$$

circle centre (1,0) radius 3

(8)



$$\text{(ii) (a)} \int_0^{\frac{\pi}{4}} (\sec x - x) \cdot (\sec x + x) dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - x^2) dx$$

$$= \left[\tan x - \frac{x^3}{3} \right]_0^{\frac{\pi}{4}}$$

$$= \left(\tan \frac{\pi}{4} - \frac{\pi^3}{192} \right) - (\tan 0 - 0)$$

$$= 1 - \frac{\pi^3}{192}$$

$$\text{(b)} \int \frac{x}{\sqrt{1-49x^2}} dx$$

$$\text{let } u = 7x^2$$

$$du = 14x dx$$

$$\int = \frac{1}{14} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{14} \sin^{-1} \frac{u}{3} + C$$

$$= \frac{1}{14} \sin^{-1} \frac{7x^2}{3} + C$$

(9)

$$(12) \quad n = 2.$$

$$8^2 + 3^0$$

$$= 64 + 1$$

$$= 65$$

which is divisible by 5

so result is true for $n = 2$.

Assume result is true for $n = k$

i.e. $8^k + 3^{k-2}$ is divisible by 5

so $8^k + 3^{k-2} = 5n$ for some $n \in \mathbb{Z}$. (*)

Consider $n = k+1$

$$\begin{aligned} & 8^{k+1} + 3^{(k+1)-2} \\ &= 8(8^k) + 3^{k-1} \end{aligned}$$

from (*)

$$8^k = 5n - 3^{k-2}$$

$$\begin{aligned} &= 8(5n - 3^{k-2}) + 3^{k-1} \\ &= 40n - 8 \times 3^{k-2} + 3^{k-1} \\ &= 40n - 3^{k-2}(8-3) \\ &= 40n - 5 \times 3^{k-2} \\ &= 5(8n - 3^{k-2}) \end{aligned}$$

which is divisible by 5.

Hence result is true for $n = k \Rightarrow$ result is true for $n = k+1$
 and since result is true for $n = 2$, by induction it must
 be true $\forall n \geq 2$.

(10)

$$\textcircled{B} \quad a, \frac{1}{4}, 1$$

$$\frac{1}{a} - a = 1 - 1 \\ a.$$

$$1 - a^2 = a - 1$$

$$a^2 + a - 2 = 0$$

$$(a+2)(a-1) = 0$$

$$a = -2 \text{ or } a = 1$$

Since $a < 0$ $a = -2$.

$$\text{common difference} = 1 + \frac{1}{2}$$

$$= 1\frac{1}{2}$$

$$= \frac{3}{2}$$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$\frac{n}{2} \left(-4 + (n-1)\frac{3}{2} \right) > 1000$$

$$n(-4 + -2n + \frac{3n}{4}(n-1)) > 1000$$

$$-8n + 3n^2 - 3n > 4000$$

$$3n^2 - 11n - 4000 > 0$$

$$\text{Sketch } y = 3n^2 - 11n - 4000$$

$$\text{Cub n-axis } y = 0$$

$$3n^2 - 11n - 4000 = 0$$

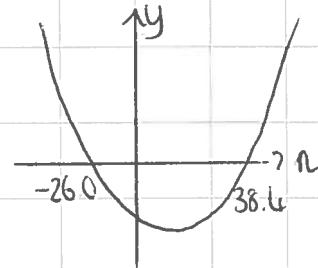
$$n = \frac{11 \pm \sqrt{121 - 4 \times 3 \times (-4000)}}{6}$$

(11)

$$n = 11 \pm \sqrt{18121}$$

6

$$n = 38.4 \text{ or } n = -26.0$$



So $n > 38.4$

smallest value for n is $n = 39$

(4) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12$

auxiliary equation

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m=2 \text{ or } m=-1$$

CF

$$y = Ae^{2x} + Be^{-x}$$

PI

Try $y = Ae^x + B$

$$\frac{dy}{dx} = Ae^x$$

$$\frac{d^2y}{dx^2} = Ae^x$$

Substitute

$$Ae^x - Ae^x - 2(Ae^x + B) = e^x + 12$$

$$-2Ae^x - 2B = e^x + 12$$

(12)

Equate

$$-2A = 1$$

$$-2B = 12$$

$$A = -\frac{1}{2}$$

$$B = -6.$$

$$\text{PII: } y = -\frac{1}{2}e^x - 6$$

General solution

$$y = \text{CF} + \text{PI}$$

$$y = Ae^{2x} + Be^{-x} - \frac{1}{2}e^x - 6.$$

$$\text{When } x=0 \quad y = -\frac{3}{2}$$

$$-\frac{3}{2} = A + B - \frac{1}{2} - 6$$

$$A + B = 5 \quad \dots \quad (1)$$

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x} - \frac{1}{2}e^x$$

$$\text{When } x=0 \quad \frac{dy}{dx} = \frac{1}{2}$$

$$\frac{1}{2} = 2A - B - \frac{1}{2}$$

$$2A - B = 1 \quad \dots \quad (2)$$

$$(1) + (2) \quad 3A = 6$$

$$A = 2$$

$$\Rightarrow B = 3.$$

Solution

$$y = 2e^{2x} + 3e^{-x} - \frac{1}{2}e^x - 6$$

$$\textcircled{15} \quad (a) \quad l_1 \quad \frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \quad (= t)$$

$$x = kt + 1$$

$$y = -t$$

$$z = t - 3$$

$$l_2 \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2} \quad (= s)$$

$$x = s + 4$$

$$y = s - 3$$

$$z = 2s - 3$$

Intersección

$$kt + 1 = s + 4$$

$$-t = s - 3$$

$$t - 3 = 2s - 3$$

- ①

solve - ②

- ③

add ② + ③

$$-3 = 3s - 6$$

$$3s = 3$$

$$s = 1$$

$$\Rightarrow -\underline{t} = 1 - 3$$

$$\underline{t = 2}$$

$$\ln \textcircled{1} \quad 2k + 1 = 1 + 4$$

$$2k = 4$$

$$\underline{\underline{k = 2}}$$

point $(5, -2, -1)$

(b) direction of L_1 $\underline{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ direction of L_2 $\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$\underline{a} \cdot \underline{b} = 2 - 1 + 2 \\ = 3$$

$$|\underline{a}| = \sqrt{6}$$

$$|\underline{b}| = \sqrt{6}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} \\ = \frac{3}{\sqrt{6} \cdot \sqrt{6}}$$

$$\cos\theta = \frac{1}{2}$$

$$\underline{\theta = 60^\circ}$$

$$(10) \quad I_n = \int_0^1 \frac{1}{(1+x^2)^n} dx \quad n \geq 1$$

$$\begin{aligned}
 (a) \quad I_n &= \int_0^1 \frac{1}{(1+x^2)^n} dx \\
 &= \int_0^1 1 \cdot (1+x^2)^{-n} dx \\
 &= \left[(1+x^2)^{-n} \cdot x \right]_0^1 - \int_0^1 x \cdot -n(1+x^2)^{-n-1} \cdot 2x dx \\
 &= (2^{-n} - 0) + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx \\
 &= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx
 \end{aligned}$$

as required

$$(b) \quad \frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

$$\frac{A(1+x^2) + B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

$$A(1+x^2) + B = x^2$$

$$\text{coeff. } x^2: \quad A = 1$$

$$\text{constant: } \quad A+B=0 \quad \Rightarrow \quad B=-1$$

$$\text{So } \frac{x^2}{(1+x^2)^{n+1}} = \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \quad (*)$$

$$\begin{aligned}
 I_{n+1} &= \int_0^1 \frac{1}{(1+x^2)^{n+1}} dx \\
 &= \int_0^1 \left(\frac{1}{(1+x^2)^n} - \frac{x^2}{(1+x^2)^{n+1}} \right) dx \quad \text{From } (*) \\
 &= \int_0^1 \frac{1}{(1+x^2)^n} dx - \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx \\
 &= I_n - \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx - (+)
 \end{aligned}$$

$$\text{From } I_n = \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx$$

$$\begin{aligned}
 I_n - \frac{1}{2^n} \cancel{\frac{2^n I_n}{2^n}} &= 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx \\
 \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx &= \frac{I_n}{2^n} - \frac{1}{2^n(2n)}
 \end{aligned}$$

$$\begin{aligned}
 \text{In (+)} \quad I_{n+1} &= I_n - \frac{I_n}{2^n} + \frac{1}{2^n(2n)} \\
 &= \frac{2n}{2n} I_n - \frac{I_n}{2^n} + \frac{1}{n \times 2^{n+1}}
 \end{aligned}$$

$$= \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n} \right) I_n \text{ as required.}$$

$$\begin{aligned}
 (c) \quad \int_0^1 \frac{1}{(1+x^2)^3} dx &= T_3 \\
 &= \frac{1}{2 \times 2^3} + \left(\frac{4-1}{4} \right) I_2 \\
 &= \frac{1}{16} + \frac{3}{4} \left(\frac{1}{4} + \frac{1}{2} T_1 \right)
 \end{aligned}$$

$$\begin{aligned}
 I_1 &= \int_0^1 \frac{1}{1+x^2} dx \\
 &= \left[\tan^{-1} x \right]_0^1 \\
 &= \tan^{-1} 1 - \tan^{-1} 0 \\
 &= \frac{\pi}{4}.
 \end{aligned}$$

$$\begin{aligned}
 \text{So } \int_0^1 \frac{1}{(1+x^2)^3} dx &= \frac{1}{16} + \frac{3}{4} \left(\frac{1}{4} + \frac{1}{2} \left(\frac{\pi}{4} \right) \right) \\
 &= \frac{4}{16} + \frac{\pi}{8} \\
 &= \frac{1}{4} + \frac{\pi}{8}
 \end{aligned}$$