

**X100/701**

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NATIONAL  
QUALIFICATIONS  
2010

FRIDAY, 21 MAY  
1.00 PM – 4.00 PM

MATHEMATICS  
ADVANCED HIGHER

**Read carefully**

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



**Answer all the questions.**

- 1.** Differentiate the following functions.

(a)  $f(x) = e^x \sin x^2$ .

3

(b)  $g(x) = \frac{x^3}{(1 + \tan x)}$ .

3

- 2.** The second and third terms of a geometric series are  $-6$  and  $3$  respectively.

Explain why the series has a sum to infinity, and obtain this sum.

5

- 3.** (a) Use the substitution  $t = x^4$  to obtain  $\int \frac{x^3}{1+x^8} dx$ .

3

- (b) Integrate  $x^2 \ln x$  with respect to  $x$ .

4

- 4.** Obtain the  $2 \times 2$  matrix  $M$  associated with an enlargement, scale factor  $2$ , followed by a clockwise rotation of  $60^\circ$  about the origin.

4

- 5.** Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer  $n$  is greater than or equal to  $3$ .

4

- 6.** Given  $\mathbf{u} = -2\mathbf{i} + 5\mathbf{k}$ ,  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ .

Calculate  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .

4

7. Evaluate

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)} dx$$

expressing your answer in the form  $\ln \frac{a}{b}$ , where  $a$  and  $b$  are integers.

6

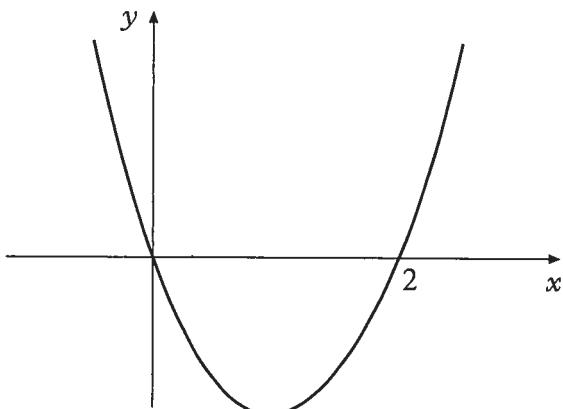
8. (a) Prove that the product of two odd integers is odd. 2

(b) Let  $p$  be an odd integer. Use the result of (a) to prove by induction that  $p^n$  is odd for all positive integers  $n$ .

4

9. Obtain the first three non-zero terms in the Maclaurin expansion of  $(1 + \sin^2 x)$ .

4

10. The diagram below shows part of the graph of a function  $f(x)$ . State whether  $f(x)$  is odd, even or neither. Fully justify your answer.

3

11. Obtain the general solution of the equation

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0.$$

4

Hence obtain the solution for which  $y = 3$  when  $x = 0$  and  $y = e^{-\pi}$  when  $x = \frac{\pi}{2}$ .

3

Marks

12. Prove by contradiction that if  $x$  is an irrational number, then  $2 + x$  is irrational. 4

13. Given  $y = t^3 - \frac{5}{2}t^2$  and  $x = \sqrt{t}$  for  $t > 0$ , use parametric differentiation to express  $\frac{dy}{dx}$  in terms of  $t$  in simplified form. 4

Show that  $\frac{d^2y}{dx^2} = at^2 + bt$ , determining the values of the constants  $a$  and  $b$ . 3

Obtain an equation for the tangent to the curve which passes through the point of inflexion. 3

14. Use Gaussian elimination to show that the set of equations

$$\begin{aligned}x - y + z &= 1 \\x + y + 2z &= 0 \\2x - y + az &= 2\end{aligned}$$

has a unique solution when  $a \neq 2.5$ . 5

Explain what happens when  $a = 2.5$ . 1

Obtain the solution when  $a = 3$ . 1

Given  $A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ , calculate  $AB$ . 1

Hence, or otherwise, state the relationship between  $A$  and the matrix

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}. 2$$

**15.**

A new board game has been invented and the symmetrical design on the board is made from four identical “petal” shapes. One of these petals is the region enclosed between the curves  $y = x^2$  and  $y^2 = 8x$  as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2.

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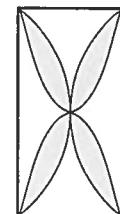
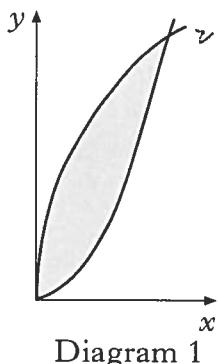


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through  $360^\circ$  about the  $y$ -axis. Find the volume of plastic required to make one counter.

5

**16.**

Given  $z = r(\cos\theta + i \sin\theta)$ , use de Moivre's theorem to express  $z^3$  in polar form.

1

Hence obtain  $\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3$  in the form  $a + ib$ .

2

Hence, or otherwise, obtain the roots of the equation  $z^3 = 8$  in Cartesian form.

4

Denoting the roots of  $z^3 = 8$  by  $z_1, z_2, z_3$ :

(a) state the value  $z_1 + z_2 + z_3$ ;

(b) obtain the value of  $z_1^6 + z_2^6 + z_3^6$ .

3

[END OF QUESTION PAPER]

# Advanced Higher 2010.

$$\textcircled{1} \quad (\text{a}) \quad f(x) = e^x \sin x^2$$

$$\begin{aligned} f'(x) &= e^x \cdot \cos x^2 \cdot 2x + \sin x^2 \cdot e^x \\ &= e^x (2x \cos x^2 + \sin x^2) \end{aligned}$$

$$(\text{b}) \quad g(x) = \frac{x^3}{1+\tan x}$$

$$\begin{aligned} g'(x) &= \frac{(1+\tan x) \cdot 3x^2 - x^3 (\sec^2 x)}{(1+\tan x)^2} \\ &= \frac{x^2 (3 + 3\tan x - x \sec^2 x)}{(1+\tan x)^2} \end{aligned}$$

\textcircled{2} Geometric series.  $a, ar, ar^2, \dots$

$$\text{So } ar = -6 \dots \textcircled{1} \text{ and } ar^2 = 3 \dots \textcircled{2}$$

$$\text{Divide. } \textcircled{2} \div \textcircled{1} \quad r = -\frac{1}{2}.$$

$$\Rightarrow -\frac{1}{2}a = -6$$

$$\underline{a = 12}$$

Sum to infinity exists since  $|r| < 1$

$$S_{\infty} = \frac{a}{1-r}$$

$$= \frac{12}{1 + \frac{1}{2}}$$

$$= 12 \times \frac{2}{3}$$

$$= \underline{\underline{8.}}$$

③ (a)  $\int \frac{x^3}{1+x^8} dx$

let  $t = x^4$

$$dt = 4x^3 dx$$

$$\int \frac{x^3}{1+x^8} dx$$

$$= \int \frac{\frac{1}{4} dt}{1+t^2}$$

$$= \frac{1}{4} \tan^{-1} t + C$$

$$= \frac{1}{4} \tan^{-1} x^4 + C.$$

(b)  $\int x^2 \ln x dx$

$$= \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

④ enlargement) scale factor 2

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

clockwise rotation  $60^\circ$

$\Rightarrow$  anticlockwise  $300^\circ$

$$\begin{pmatrix} \cos 300 & -\sin 300 \\ \sin 300 & \cos 300 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 60 & \sin 60 \\ -\sin 60 & \cos 60 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

$$\textcircled{5} \quad \binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

$$\text{LHS} \quad \binom{n+1}{3} - \binom{n}{3}$$

$$= \frac{n+1}{3} - \frac{n}{3}$$

$$= \frac{(n+1)!}{3!(n+1-3)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{(n+1)!}{3!(n-2)!} - \frac{n!(n-2)}{3!(n-2)!}$$

$$= \frac{(n+1)! - n!(n-2)}{3!(n-2)!}$$

$$= \frac{n!( (n+1) - (n-2) )}{3!(n-2)!}$$

$$= \frac{n! \cdot 3}{3!(n-2)!}$$

$$= \frac{n!}{2!(n-2)!}$$

$$= \binom{n}{2}$$

= RHS      as required.

$$\textcircled{6} \quad \underline{u} = -2\underline{i} + 5\underline{k} \quad \underline{v} = 3\underline{i} + 2\underline{j} - \underline{k}$$

$$\underline{w} = -\underline{i} + \underline{j} + 4\underline{k}$$

$$\underline{u} \cdot (\underline{v} \times \underline{w})$$

$$\begin{aligned}\underline{v} \times \underline{w} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} \\ &= \underline{i}(8+1) - \underline{j}(12-1) + \underline{k}(3+2) \\ &= 9\underline{i} - 11\underline{j} + 5\underline{k}\end{aligned}$$

$$\begin{aligned}\underline{u} \cdot (\underline{v} \times \underline{w}) &= \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -11 \\ 5 \end{pmatrix} \\ &= -18 + 25 \\ &= 7.\end{aligned}$$

$$\textcircled{7} \quad \int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)} dx$$

Write as partial fractions.

$$\begin{aligned}\frac{3x+5}{(x+1)(x+2)(x+3)} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\ &= \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}\end{aligned}$$

$$\text{So } 3x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$$\text{Let } x = -2 \quad -1 = -B \quad B = 1$$

$$\text{Let } x = -3 \quad -4 = 2C \quad C = -2$$

$$\text{Let } x = -1 \quad 2 = 2A \quad A = 1$$

$$\begin{aligned} \int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)} dx &= \int_1^2 \left( \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3} \right) dx \\ &= \left[ \ln|x+1| + \ln|x+2| - 2\ln|x+3| \right]_1^2 \\ &= (\ln 3 + \ln 4 - 2\ln 5) - (\ln 2 + \ln 3 - 2\ln 4) \\ &= \ln \frac{3 \times 4}{2 \times 5} - \ln \frac{2 \times 3}{3 \times 4} \\ &= \ln \frac{3 \times 4 \times 16^4}{2 \times 25 \times 2} \\ &= \ln \frac{48}{50} \\ &= \ln \frac{32}{25} \end{aligned}$$

⑧ (a) Let odd integers be  $2k+1$  and  $2h+1$ .

$$\begin{aligned}\text{product} &= (2k+1)(2h+1) \\ &= 4kh + 2k + 2h + 1 \\ &= 2(2kh + k + h) + 1 \\ &\text{which is odd.}\end{aligned}$$

(b)  $p = 2k+1 \quad k \in \mathbb{Z}$

Prove  $p^n = (2k+1)^n$  is true for all  $n \in \mathbb{Z}^+$

$$\begin{array}{ll} n=1 & p \\ & = 2k+1 \quad \text{which is odd} \end{array}$$

Assume result is true for  $n=r$

i.e.  $p^r$  is odd

so  $p^r = 2h+1$  for some  $h \in \mathbb{Z}$

Consider  $n=r+1$

$$\begin{aligned}p^{r+1} &= p(p^r) \\ &= (2k+1)(2h+1) \\ &= \text{product of two odd numbers} \\ &\Rightarrow \text{odd answer.}\end{aligned}$$

so true for  $n=r+1$

Since result is true for  $n=1$  and if result is true for  $n=k$  then it is also true for  $n=k+1$ , hence by induction result is true for all positive integers.

$$\textcircled{9} \quad f(x) = (1 + \sin^2 x) \quad f(0) = 1$$

$$f'(x) = 2\sin x \cdot \cos x \quad f'(0) = 0$$

$$= \sin 2x$$

$$f''(x) = 2\sin x \cdot 2\cos 2x \quad f''(0) = 2$$

$$f'''(x) = -4\sin 2x \quad f'''(0) = 0$$

$$P^{(v)}(x) = -8 \cos 2x \quad P^{(v)}(0) = -8$$

$$(1 + \sin^2 x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$= 1 + \frac{x^2}{2!} (2) + \frac{x^4}{4!} (-8)$$

$$= 1 + x^2 - \frac{1}{3} x^4 + \dots$$

\textcircled{10} Neither odd or even.

Odd functions have half turn symmetry about 0

Even functions are symmetrical about y-axis.

$$\textcircled{11} \quad \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0$$

auxiliary equation

$$m^2 + 4m + 5 = 0$$

$$m = -4 \pm \sqrt{16 - 4 \times 5}$$

$$m = \frac{-4 \pm 2i}{2}$$

$$m = -2 \pm i$$

General solution

$$y = Ae^{-2x} \sin x + Be^{-2x} \cos x.$$

$$y=3 \text{ when } x=0 \Rightarrow 3 = B$$

$$y = e^{-\pi} \text{ when } x = \frac{\pi}{2} \Rightarrow e^{-\pi} = Ae^{-\pi} \sin \frac{\pi}{2} + 0.$$

$$e^{-\pi} = Ae^{-\pi}$$
$$A = 1$$

$$\text{Solution } y = e^{-2x} \sin x + 3e^{-2x} \cos x.$$

(12) If  $x$  is an irrational number assume that  $2+x$  is rational

$$\text{so } 2+x = \frac{m}{n} \text{ for some } m, n \in \mathbb{Z}$$

$$x = \frac{m}{n} - 2$$

$$x = \frac{m-2n}{n}$$

which is rational.

This is a contradiction so  $2+x$  must be irrational.

(B)

$$y = t^3 - \frac{5}{2}t^2$$

$$\frac{dy}{dt} = 3t^2 - 5t$$

$$x = \sqrt{t}$$

$$x = t^{\frac{1}{2}}$$

$$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= (3t^2 - 5t) \cdot 2\sqrt{t}$$

$$= 6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= (15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}) \cdot 2t^{\frac{1}{2}} \\ &= 30t^2 - 30t \\ &= at^2 + bt \quad \text{where } a = 30, b = -30\end{aligned}$$

Point of inflection

$$\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 30t^2 - 30t = 0$$

$$30t(t-1) = 0$$

$$t=0 \text{ or } t=1.$$

not possible.  
( $t > 0$ )

$$\begin{aligned}\text{When } t=1 &\quad \frac{dy}{dx} = 6-10 \\ &\quad = \underline{\underline{-4}} \\ &\quad x = \underline{\underline{1}} \quad y = 1 - \frac{5}{2} \\ &\quad = \underline{\underline{-\frac{3}{2}}}.\end{aligned}$$

Equation

$$y - b = m(x-a)$$
$$y - \left(-\frac{3}{2}\right) = -4(x-1)$$

$$2y + 3 = -8x + 8$$
$$2y + 8x = 5.$$

(14)

$$x - y + z = 1$$

$$x + y + 2z = 0$$

$$2x - y + az = 2$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array} \right)$$

$$R_2 - R_1 \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array} \right)$$

$$R_3 - 2R_1 \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 5-2a & -1 \end{array} \right)$$

If  $5-2a \neq 0$  there is a unique solution  
ie  $a \neq 2.5$

When  $a = 2.5$  equations are inconsistent and there  
is no solution

When  $a = 3$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$-z = -1$$

$$z = 1$$

$$2y + z = -1$$

$$2y - 1 = -1$$

$$2y = -2$$

$$y = -1$$

$$x - y + z = 1$$

$$x + 1 + 1 = 1$$

$$x = -1$$

Solution  $x = -1, y = -1$  and  $z = 1$

$$\begin{aligned} AB &= \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \quad \leftarrow \text{ solution} \end{aligned}$$

Equations give:

$$C \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B$$

$$CAB = B.$$

$$\text{so } CA = I$$

C is the inverse of matrix A.

(15) Intersection  $y = x^2$   $y^2 = 8x$   
 $(x^2)^2 = 8x$   
 $x^4 - 8x = 0$   
 $x(x^3 - 8) = 0$   
 $x = 0 \quad x = 2$

$$\begin{aligned} \text{Area} &= 4 \int_0^2 (\sqrt{8x} - x^2) dx \\ &= 4 \int_0^2 (2\sqrt{2} \cdot x^{1/2} - x^2) dx \\ &= 4 \left[ 2\sqrt{2} \cdot \frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^2 \\ &= 4 \left( 2\sqrt{2} \cdot \frac{2}{3} (2^{3/2}) - \frac{8}{3} \right) - 0 \\ &= 4 \left( \frac{16}{3} - \frac{8}{3} \right) \\ &= 4 \left( \frac{8}{3} \right) \\ &= \frac{32}{3}. \end{aligned}$$

$$\begin{aligned} V &= \pi \int_0^4 (y_1^2 - y_2^2) dx \\ &= \pi \int_0^4 ((\sqrt{8x})^2 - (x^2)^2) dx \\ &= \pi \int_0^4 (8x - x^4) dx \end{aligned}$$

$$\begin{aligned} \text{Revolution about } y\text{-axis} \\ V &= \pi \int_0^4 x_1^2 dy - \pi \int_0^4 x_2^2 dy \\ &= \pi \int_0^4 y dy - \pi \int_0^4 \frac{y^4}{64} dy \\ &= \pi \left[ \frac{y^2}{2} \right]^4 - \pi \left[ \frac{y^5}{320} \right]^4 \end{aligned}$$

$$\begin{aligned}
 &= \pi \left[ \frac{8x^2}{2} - \frac{x^5}{5} \right]^2 \\
 &= \pi \left( 16 - \frac{32}{5} \right) \\
 &= \frac{48\pi}{5} \\
 &= \frac{24\pi}{5}
 \end{aligned}$$

(16)  $z = r(\cos\theta + i\sin\theta)$

$$\begin{aligned}
 z^3 &= r^3 (\cos\theta + i\sin\theta)^3 \\
 &= r^3 (\cos 3\theta + i\sin 3\theta)
 \end{aligned}$$

$$\begin{aligned}
 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^3 &= \cos 2\pi + i \sin 2\pi \\
 &= 1 \quad a=1 \quad b=0.
 \end{aligned}$$

$$z^3 = 8$$

$$z^3 - 8 = 0$$

$$(z-2)(z^2+2z+4) = 0$$

$$\begin{aligned}
 z &= 2 \quad \text{or} \quad z = \frac{-2 \pm \sqrt{4-16}}{2} \\
 &= \frac{-2 \pm 2\sqrt{3}i}{2} \\
 &= -1 \pm \sqrt{3}i
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad z_1 + z_2 + z_3 &= 2 + (-1 + \sqrt{3}i) + (-1 - \sqrt{3}i) \\
 &= 0
 \end{aligned}$$

$$(b) z_1^3 = 8$$

$$z_2^3 = 8$$

$$z_3^3 = 8$$

$$\text{so } z_1^6 + z_2^6 + z_3^6$$

$$= 8^2 + 8^2 + 8^2$$

$$= 192$$