

X100/701

NATIONAL
QUALIFICATIONS
2008

TUESDAY, 20 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
3. **Full credit will be given only where the solution contains appropriate working.**

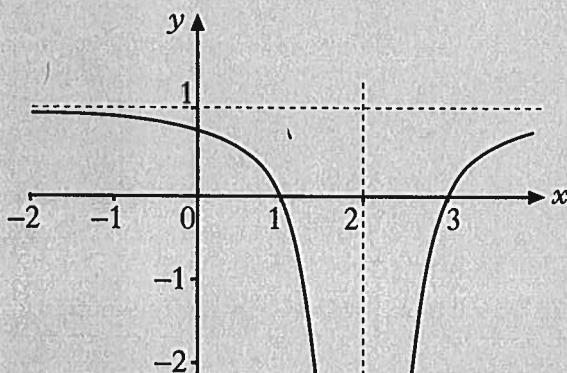


Answer all the questions.

1. The first term of an arithmetic sequence is 2 and the 20th term is 97. Obtain the sum of the first 50 terms. 4

- H/W** 2. (a) Differentiate $f(x) = \cos^{-1}(3x)$ where $-\frac{1}{3} < x < \frac{1}{3}$. 2
 (b) Given $x = 2 \sec \theta$, $y = 3 \sin \theta$, use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ . 3

3. Part of the graph $y = f(x)$ is shown below, where the dotted lines indicate asymptotes. Sketch the graph $y = -f(x + 1)$ showing its asymptotes. Write down the equations of the asymptotes. 4



- H/W** 4. Express $\frac{12x^2 + 20}{x(x^2 + 5)}$ in partial fractions. 3

Hence evaluate

$$\int_1^2 \frac{12x^2 + 20}{x(x^2 + 5)} dx. \quad 3$$

- H/W** 5. A curve is defined by the equation $xy^2 + 3x^2y = 4$ for $x > 0$ and $y > 0$.

Use implicit differentiation to find $\frac{dy}{dx}$. 3

Hence find an equation of the tangent to the curve where $x = 1$. 3

- | | Marks |
|---|-------------|
| 6. Let the matrix $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$. | |
| (a) Obtain the value(s) of x for which A is singular. | 2 |
| (b) When $x = 2$, show that $A^2 = pA$ for some constant p .
Determine the value of q such that $A^4 = qA$. | 3 |
| 7. Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$. | 5 |
|
<i>H/W</i> 8. Write down and simplify the general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.
Hence, or otherwise, obtain the term in x^{14} . | 3
2 |
|
<i>H/W</i> 9. Write down the derivative of $\tan x$.
Show that $1 + \tan^2 x = \sec^2 x$.
Hence obtain $\int \tan^2 x \, dx$. | 1
1
2 |
| 10. A body moves along a straight line with velocity $v = t^3 - 12t^2 + 32t$ at time t . | |
| (a) Obtain the value of its acceleration when $t = 0$. | 1 |
| (b) At time $t = 0$, the body is at the origin O . Obtain a formula for the displacement of the body at time t .
Show that the body returns to O , and obtain the time, T , when this happens. | 2
2 |
| 11. For each of the following statements, decide whether it is true or false and prove your conclusion.
A For all natural numbers m , if m^2 is divisible by 4 then m is divisible by 4.
B The cube of any odd integer p plus the square of any even integer q is always odd. | 5 |
| 12. Obtain the first three non-zero terms in the Maclaurin expansion of $x \ln(2+x)$.
Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x \ln(2-x)$.
Hence obtain the first two non-zero terms in the Maclaurin expansion of $x \ln(4-x^2)$. | 3
2
2 |

[Throughout this question, it can be assumed that $-2 < x < 2$.]

[Turn over for Questions 13 to 16 on Page four]

13. Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2.$$

7

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = 1$, when $x = 0$, find the particular solution.

3

14. (a) Find an equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$ and $C(0, 3, 3)$.

3

- (b) The plane π_2 has equation $x + 3y - z = 2$.

Given that the point $(0, a, b)$ lies on both the planes π_1 and π_2 , find the values of a and b . Hence find an equation of the line of intersection of the planes π_1 and π_2 .

4

- (c) Find the size of the acute angle between the planes π_1 and π_2 .

3

15. Let $f(x) = \frac{x}{\ln x}$ for $x > 1$.

- (a) Derive expressions for $f'(x)$ and $f''(x)$, simplifying your answers.

2,2

- (b) Obtain the coordinates and nature of the stationary point of the curve $y = f(x)$.

3

- (c) Obtain the coordinates of the point of inflexion.

2

16. Given $z = \cos \theta + i \sin \theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ , where k is a positive integer.

Hence show that $\frac{1}{z^k} = \cos k\theta - i \sin k\theta$.

3

Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of z .

2

Show that $\cos^2 \theta \sin^2 \theta = -\frac{1}{16} \left(z^2 - \frac{1}{z^2} \right)^2$.

3

Hence show that $\cos^2 \theta \sin^2 \theta = a + b \cos 4\theta$, for suitable constants a and b .

2

[END OF QUESTION PAPER]

Advanced Higher 2008

$$\textcircled{1} \quad a = 2 \\ u_{20} = 97$$

$$u_n = a + (n-1)d$$

$$u_{20} = 2 + 19d$$

$$97 = 2 + 19d$$

$$19d = 95$$

$$d = 5$$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$S_{50} = \frac{50}{2} (4 + 49 \times 5)$$

$$S_{50} = 6225$$

$$\textcircled{2} \quad (\text{a}) \quad f(x) = \cos^{-1} 3x$$

$$f'(x) = -\frac{1}{\sqrt{1-(3x)^2}} \times 3$$

$$= \frac{-3}{\sqrt{1-9x^2}}$$

$$\text{(b)} \quad x = 2\cos\theta$$

$$= 2(\cos\theta)^{-1}$$

$$\frac{dx}{d\theta} = -2(\cos\theta)^{-2} \cdot -\sin\theta$$

$$= \frac{2\sin\theta}{\cos^2\theta}$$

$$y = 3\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

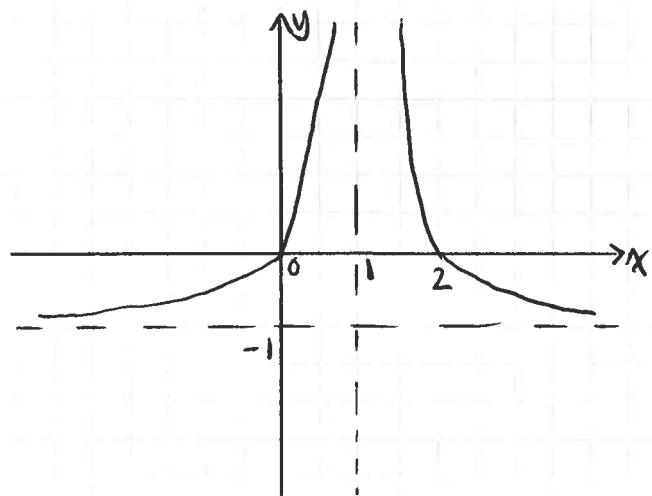
$$= 3\cos\theta \cdot \frac{\cos^2\theta}{2\sin\theta}$$

$$= \frac{3}{2} \cos^3\theta \cot\theta$$

(2)

③ $y = -f(x+1)$
 ↗
 shift to left
 reflected in x -axis

asymptotes $x=1$
 $y=-1$



④ $\frac{12x^2 + 20}{x(x^2 + 5)}$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+5}$$

$$= \frac{A(x^2+5) + Bx^2+Cx}{x(x^2+5)}$$

identity $12x^2 + 20 = Ax^2 + 5A + Bx^2 + Cx$

$$x=0 \quad 20 = 5A$$

$$A=4$$

coeff of x $0 = C$

coeff of x^2 $12 = 4 + B$

$$B=8$$

$$\frac{12x^2 + 20}{x(x^2 + 5)} = \frac{4}{x} + \frac{8x}{x^2 + 5}$$

$$\begin{aligned} \int_1^2 \frac{12x^2 + 20}{x(x^2 + 5)} dx &= \int_1^2 \left(\frac{4}{x} + \frac{8x}{x^2 + 5} \right) dx \\ &= \int_1^2 \left(\frac{4}{x} + 4 \cdot \frac{(2x)}{x^2 + 5} \right) dx \\ &= \left[4 \ln|x| + 4 \ln|x^2 + 5| \right]_1^2 \\ &= 4 \ln 2 + 4 \ln 9 - (0 + 4 \ln 6) \end{aligned}$$

(3)

$$= 4(\ln 2 + \ln 9 - \ln 6) \\ = 4 \ln 3.$$

⑤ $xy^2 + 3x^2y = 4$

$$x \cdot 2y \frac{dy}{dx} + y^2 + 3x^2 \frac{dy}{dx} + y \cdot 6x = 0$$

$$2xy \frac{dy}{dx} + 3x^2 \frac{dy}{dx} = -6xy - y^2$$

$$\frac{dy}{dx} = \frac{-6xy - y^2}{2xy + 3x^2}$$

When $x = 1$ in $xy^2 + 3x^2y = 4$

$$y^2 + 3y - 4 = 0$$

$$(y+4)(y-1) = 0$$

$$y = -4 \text{ or } y = 1$$

$$\text{Since } y > 0 \Rightarrow y = 1$$

$$\ln \frac{dy}{dx} = \frac{-6xy - y^2}{2xy + 3x^2} \\ = \frac{-6 - 1}{2 + 3} \\ = -\frac{7}{5}$$

Equation

$$y - b = m(x - a) \\ y - 1 = -\frac{7}{5}(x - 1) \\ 5y - 5 = -7x + 7 \\ \underline{\underline{5y + 7x = 12}}.$$

⑥ $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$

(a) singular $\Rightarrow \det A = 0$

$$4 - x^2 = 0$$

$$x = \pm 2$$

$$(b) \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix}$$

$$= 5 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= 5A$$

so $A^2 = pA$ where $p=5$.

$$A^4 = A^2 \cdot A^2$$

$$= 5A \cdot 5A$$

$$= 25A^2$$

$$= 25(5A)$$

$$= 125A$$

$$q = 125.$$

$$(7) \quad \int 8x^2 \sin 4x \, dx$$

$$\begin{aligned} &= 8x^2 \cdot -\frac{1}{4} \cos 4x + \frac{1}{4} \int 16x \cos 4x \, dx \\ &= -2x^2 \cos 4x + \frac{1}{4} \left(16x \cdot \frac{1}{4} \sin 4x - \int \frac{1}{4} \sin 4x \cdot 16 \, dx \right) \\ &= -2x^2 \cos 4x + \cancel{16x} \sin 4x + \cancel{\frac{1}{4}} \cdot \frac{1}{4} \cos 4x + C \\ &= -2x^2 \cos 4x + \cancel{16x} \sin 4x + \frac{1}{4} \cos 4x + C. \end{aligned}$$

$$(8) \quad \left(x^2 + \frac{1}{x} \right)^{10}$$

general term

$$\begin{aligned} & {}^{10}C_r (x^2)^{10-r} \left(\frac{1}{x}\right)^r \\ &= {}^{10}C_r x^{20-2r} x^{-r} \\ &= {}^{10}C_r x^{20-3r} \end{aligned}$$

term in x^{14} when $20 - 3r = 14$
 $r = 2.$

$$\begin{aligned} & {}^{10}C_2 x^{14} \\ &= 45x^{14} \end{aligned}$$

(9) $\frac{d}{dx}(\tan x) = \sec^2 x$

We know $\sin^2 x + \cos^2 x = 1$

divide by $\cos^2 x$ $\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$
 $\tan^2 x + 1 = \sec^2 x.$

as required.

$$\begin{aligned} \int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C. \end{aligned}$$

(10) (a) $v = t^3 - 12t^2 + 32t$

$$a = \frac{dv}{dt} = 3t^2 - 24t + 32.$$

When $t = 0$ $a = 32.$

(b) $\frac{ds}{dt} = v = t^3 - 12t^2 + 32t$

$$s = \int (t^3 - 12t^2 + 32t) dt$$

$$s = \frac{t^4}{4} - 4t^3 + 16t^2 + C$$

When $t = 0, s = 0$ $0 = 0 - 0 + 0 + C$
 $C = 0.$

$$s(t) = \frac{t^4}{4} - 4t^3 + 16t^2$$

(6)

Returns to 0 $\Rightarrow s=0$

$$\frac{t^4}{4} - 4t^3 + 16t^2 = 0$$

$$t^4 - 16t^3 + 64t^2 = 0$$

$$t^2(t^2 - 16t + 64) = 0$$

$$t^2(t-8)(t+8) = 0$$

$$t=0 \text{ or } t=8$$

The body returns to 0 when $t=8$

(11) A False $m=2$

$m^2=4$ which is divisible by 4

but m is not divisible by 4.

B. True odd number $(2k+1)$

even number $2k$

Consider $(2k+1)^3 + (2k)^3$ square.

$$\begin{aligned}
 &= (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 8k^3 \\
 &= 8k^3 + 12k^2 + 6k + 8k^2 + 1 \\
 &= 2(4k^3 + 6k^2 + 3k + 4k^3) + 1 \\
 &= 2r + 1
 \end{aligned}$$

where $r = 4k^3 + 6k^2 + 3k + 4k^3$.

= odd

\therefore result is true.

(12)

Consider $f(x) = \ln(2+x)$

$f(0) = \ln 2$

$$f'(x) = \frac{1}{2+x}$$

$$f'(0) = \frac{1}{2}$$

$$f''(x) = -(2+x)^{-2}$$

$$= -\frac{1}{(2+x)^2}$$

$$f''(0) = -\frac{1}{4}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0)$$

$$= \ln 2 + \frac{1}{2}x + \frac{x^2}{2} \left(-\frac{1}{4}\right)$$

$$\ln(2+x) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 \quad (\star)$$

$$\text{So } x \ln(2+x) = x \ln 2 + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \dots$$

Substitute $-x$ for x in (\star)

$$\ln(2-x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$\text{So } x \ln(2-x) = x \ln 2 - \frac{1}{2}x^2 - \frac{1}{8}x^3 + \dots$$

$$\begin{aligned} x \ln(4-x^2) &= x \ln(2-x)(2+x) \\ &= x (\ln(2-x) + \ln(2+x)) \\ &= x \ln(2-x) + x \ln(2+x) \\ &= x \ln 2 - \frac{1}{2}x^2 - \frac{1}{8}x^3 + x \ln 2 + \frac{1}{2}x^2 - \frac{1}{8}x^3 \\ &= 2x \ln 2 - \frac{1}{4}x^3 + \dots \end{aligned}$$

$$(13) \quad \frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 2x^2.$$

auxiliary equation

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m=2 \text{ or } m=1$$

CF.

$$y = Ae^{2x} + Be^x$$

PI

Try

$$y = Cx^2 + Dx + E$$

$$\frac{dy}{dx} = 2Cx + D$$

$$\frac{d^2y}{dx^2} = 2C.$$

Substitute

$$2C - 6Cx - 3D + 2Cx^2 + 2Dx + 2E = 2x^2$$

(x^2)

$$2C = 2$$

$$C = 1$$

x

$$-6C + 2D = 0$$

$$-6 + 2D = 0$$

$$D = 3$$

constant

$$2C - 3D + 2E = 0$$

$$2 - 9 + 2E = 0$$

$$E = \frac{7}{2}$$

General solution

$$y = Ae^{2x} + Be^x + x^2 + 3x + \frac{7}{2}$$

$$y = \frac{1}{2} \text{ when } x = 0 \Rightarrow \frac{1}{2} = A + B + \frac{7}{2}$$

$$A+B = -3 \dots \textcircled{1}$$

$$\frac{dy}{dx} = 1 \text{ when } x=0$$

$$\frac{dy}{dx} = 2Ae^{2x} + Be^x + 2x + 3$$

$$1 = 2A + B + 3$$

$$2A + B = -2 \dots \textcircled{2}$$

$$\textcircled{1} - \textcircled{2}$$

$$-A = 5 - 1$$

$$A = -4$$

$$\Rightarrow B = -4.$$

$$y = -4e^{2x} - \frac{4}{3}e^x + x^2 + 3x + \frac{7}{2}.$$

(14)

$$\begin{aligned} \textcircled{14} \quad \vec{AB} &= \underline{b} - \underline{a} \\ &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{BC} &= \underline{c} - \underline{b} \\ &= \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \end{aligned}$$

normal vector $\begin{vmatrix} i & j & k \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & 4 & 2 \end{vmatrix}$
 $= -4\underline{i} - 2\underline{j}$

Plane $-4x - 2y = k$

Substitute $(1, 1, 1)$ $-4 - 2 = k$
 $k = -6$

$$\begin{aligned} -4x - 2y &= -6 \\ 4x + 2y &= 6. \quad (\Pi_1) \end{aligned}$$

(b) Substitute $(0, a, b)$ in Π_1 , $0 + 2a = 6$
 $a = 3$

Substitute $(0, a, b)$ in Π_2 $0 + 3a - b = 2$
 $9 - b = 2$
 $b = 7.$

point $(0, 3, 7)$

Direction vector $\begin{vmatrix} i & j & k \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 3 & -1 & 1 \end{vmatrix}$
 $= 2\underline{i} - 4\underline{j} - 10\underline{k}$ point $(0, 3, 7)$

Line. $\frac{x}{2} = \frac{y-3}{-4} = \frac{z-7}{-10} (=t)$

(c) angle between normals.

$$\underline{n}_1 = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix}$$

$$\underline{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\underline{n}_1 \cdot \underline{n}_2 = -4 + (-6) + 0 \\ = -10$$

$$|\underline{n}_1| = \sqrt{16+4} \\ = \sqrt{20}$$

$$|\underline{n}_2| = \sqrt{1+9+1} \\ = \sqrt{11}$$

$$\cos\theta = \frac{-10}{\sqrt{20} \sqrt{11}}$$

$$\theta = 132.4^\circ$$

$$\text{acute angle} = 180 - 132.4 \\ = \underline{\underline{47.6^\circ}}$$

$$(15) \quad f(x) = \frac{x}{\ln x}$$

$$(a) \quad f'(x) = \frac{\ln x \cdot 1 - x \cdot \frac{1}{x}}{(\ln x)^2} \\ = \frac{\ln x - 1}{(\ln x)^2}$$

$$f''(x) = \frac{(\ln x)^2 \cdot \frac{1}{x} - (\ln x - 1) \cdot 2\ln x \cdot \frac{1}{x}}{(\ln x)^4} \\ = \frac{(\ln x)^2 - 2(\ln x)^2 + 2\ln x}{x(\ln x)^4} \\ = \frac{2\ln x - (\ln x)^2}{x(\ln x)^4}$$

$$(b) \quad \text{Stationary points} \quad f'(x) = 0$$

$$\frac{\ln x - 1}{(\ln x)^2} = 0$$

$$\ln x = 1$$

$$x = e$$

$$\Rightarrow f(e) = e \quad \text{point } (e, e)$$

When $x = e$

$$f''(e) = \frac{2\ln e - (\ln e)^2}{e(\ln e)^4} \\ = \frac{2-1}{e^4} \\ = \frac{1}{e^4} > 0 \quad \text{so } (e, e) \text{ is a minimum T.P.}$$

$$(c) \quad \text{Point of inflection} \quad f''(x) = 0$$

$$2\ln x - (\ln x)^2 = 0$$

$$(\ln x)(2 - \ln x) = 0$$

$$\text{not possible} \Rightarrow \ln x = 0 \quad \text{or} \quad 2 - \ln x = 0$$

$$\text{so } \ln x = 2.$$

$$x = e^2$$

$$y = \frac{e^2}{\ln e^2}$$

$$y = \frac{1}{2}e^2$$

point $(e^2, \frac{1}{2}e^2)$

(16)

$$z = \cos\theta + i\sin\theta$$

$$z^k = (\cos\theta + i\sin\theta)^k$$

$$= \cos k\theta + i\sin k\theta \dots \textcircled{1}$$

Replace k by $-k$.

$$z^{-k} = \cos(-k\theta) + i\sin(-k\theta)$$

$$\frac{1}{z^k} = \cos k\theta - i\sin k\theta \dots \textcircled{2}$$

Add $\textcircled{1}$ and $\textcircled{2}$

$$z^k + \frac{1}{z^k} = 2\cos k\theta$$

$$\cos k\theta = \frac{1}{2} (z^k + \frac{1}{z^k})$$

Subtract $\textcircled{1}$ - $\textcircled{2}$

$$z^k - \frac{1}{z^k} = 2i\sin k\theta.$$

$$\sin k\theta = \frac{1}{2i} (z^k - \frac{1}{z^k})$$

$$\text{When } k=1 \quad \cos\theta = \frac{1}{2} (z + \frac{1}{z}) \quad \sin\theta = \frac{1}{2i} (z - \frac{1}{z})$$

$$\cos\theta \sin\theta = \frac{1}{4i} (z + \frac{1}{z})(z - \frac{1}{z})$$

$$\cos\theta \sin\theta = \frac{1}{4i} (z^2 - \frac{1}{z^2})$$

$$\cos^2\theta \sin^2\theta = -\frac{1}{16} (z^2 - \frac{1}{z^2})^2 \quad \text{as required.}$$

$$z^2 = \cos 2\theta + i \sin 2\theta \quad (\text{from ①})$$

$$\frac{1}{z^2} = \cos 2\theta - i \sin 2\theta. \quad (\text{from ②})$$

$$\begin{aligned} z^2 - \frac{1}{z^2} &= \cos 2\theta + i \sin 2\theta - \cos 2\theta + i \sin 2\theta \\ &= 2i \sin 2\theta. \end{aligned}$$

$$\cos^2 \theta \sin^2 \theta = -\frac{1}{16} (2i \sin 2\theta)^2$$

$$= -\frac{1}{16} (-4 \sin^2 2\theta)$$

$$= \frac{1}{4} \sin^2 2\theta$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 4\theta)$$

$$= \frac{1}{8} - \frac{1}{8} \cos 4\theta$$

as required.

$$a = \frac{1}{8} \quad b = -\frac{1}{8}$$