X100/701

NATIONAL
QUALIFICATIONS
2007

TUESDAY, 15 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**

SCOTTISH
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AUTHORITY

Marks

Answer all the questions.

1. Express the binomial expansion of $\left(x - \frac{2}{x}\right)^4$ in the form $ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$ for integers a, b, c, d and e. **4**

2. Obtain the derivative of each of the following functions:

   (*a*) $f(x) = \exp(\sin 2x)$; **3**

   (*b*) $y = 4^{(x^2+1)}$. **3**

3. Show that $z = 3 + 3i$ is a root of the equation $z^3 - 18z + 108 = 0$ and obtain the remaining roots of the equation. **4**

4. Express $\dfrac{2x^2 - 9x - 6}{x(x^2 - x - 6)}$ in partial fractions. **3**

   Given that

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}\,dx = \ln\frac{m}{n},$$

   determine values for the integers m and n. **3**

5. Matrices A and B are defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

   (*a*) Find the product AB. **2**

   (*b*) Obtain the determinants of A and of AB. **2**

   Hence, or otherwise, obtain an expression for det B. **1**

6. Find the Maclaurin series for $\cos x$ as far as the term in x^4. **2**

   Deduce the Maclaurin series for $f(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4. **2**

   Hence write down the first three non-zero terms of the series for $f(3x)$. **1**

Marks

7. Use the Euclidean algorithm to find integers p and q such that $599p + 53q = 1$. **4**

8. Obtain the general solution of the equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$. **6**

9. Show that $\sum_{r=1}^{n}(4-6r) = n - 3n^2$. **2**

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$. **1**

Show that $\sum_{r=q+1}^{2q}(4-6r) = q - 9q^2$. **2**

10. Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$. **5**

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{(1+x^2)^2}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Write down the volume of this solid. **1**

11. Given that $|z-2| = |z+i|$, where $z = x + iy$, show that $ax + by + c = 0$ for suitable values of a, b and c. **3**

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2| = |z+i|$. **1**

12. Prove by induction that for $a > 0$,

$$(1 + a)^n \geq 1 + na$$

for all positive integers n. **5**

13. A curve is defined by the parametric equations $x = \cos 2t$, $y = \sin 2t$, $0 < t < \frac{\pi}{2}$.

(*a*) Use parametric differentiation to find $\frac{dy}{dx}$.
Hence find the equation of the tangent when $t = \frac{\pi}{8}$. **5**

(*b*) Obtain an expression for $\frac{d^2y}{dx^2}$ and hence show that $\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = k$, where k is an integer. State the value of k. **5**

[Turn over for Questions 14 to 16 on *Page four*

Marks

14. A garden centre advertises young plants to be used as hedging.

After planting, the growth G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.

(*a*) Express G in terms of t and k. **4**

(*b*) Given that a plant grows 0·6 metres by the end of 5 years, find the value of k correct to 3 decimal places. **2**

(*c*) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? **2**

(*d*) Given that the initial height of the plants was 0·3 m, what is the likely long-term height of the plants? **2**

15. Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x = 2 + s,\ y = -s,\ z = 2 - s \qquad L_2 : x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

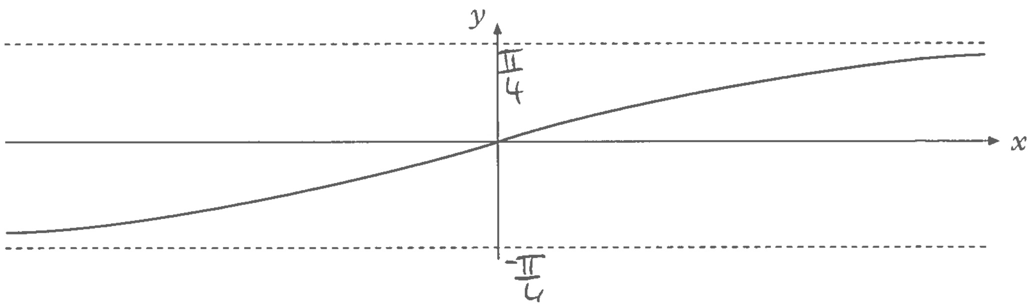
(*a*) Show that L_1 and L_2 do not intersect. **3**

(*b*) The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3. **3**

(*c*) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1. **3**

(*d*) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ. **1**

16.



(*a*) The diagram shows part of the graph of $f(x) = \tan^{-1} 2x$ and its asymptotes. State the equations of these asymptotes. **2**

(*b*) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$, $x = \frac{1}{2}$. **5**

(*c*) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x-axis and the lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$. **3**

[END OF QUESTION PAPER]

Advanced Higher Mathematics 2007

1.

$$\left(x - \frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$$ **1 for powers**
1 for coeffs

$$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$$ **2E1**

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2. (a)

$$f(x) = \exp(\sin 2x)$$

$$f'(x) = 2\cos 2x \exp(\sin 2x)$$ **M1,2E1**

(b)

$$y = 4^{(x^2+1)}$$

$$\ln y = \ln\left(4^{(x^2+1)}\right) = \left(x^2 + 1\right)\ln 4$$ **M1**

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln 4$$ **1**

$$\frac{dy}{dx} = 2x\ln 4 \,.\, 4^{(x^2+1)}$$ **1**

Alternative:

$$y = 4^{(x^2+1)}$$

$$4 = e^{\ln 4}$$ **1**

$$y = e^{\ln 4(x^2+1)}$$

$$\frac{dy}{dx} = \ln 4\, 2x\, e^{\ln 4(x^2+1)}$$ **1,1**

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3. $(3 + 3i)^3 = 27 + 81i + 81i^2 + 27i^3 = -54 + 54i$. Thus

$$(3 + 3i)^3 - 18(3 + 3i) + 108 =$$

$$-54 + 54i - 54 - 54i + 108 = 0$$ **1**

Since $3 + 3i$ is a root, $3 - 3i$ is a root. **1**

These give a factor $(z - (3 + 3i))(z - (3 - 3i)) = (z - 3)^2 + 9 = z^2 - 6z + 18$. **1**

$$z^3 - 18z + 108 = (z^2 - 6z + 18)(z + 6)$$

The remaining roots are $3 - 3i$ and -6. **1**

4.

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{2x^2 - 9x - 6}{x(x + 2)(x - 3)} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 3}$$ **1**

$$2x^2 - 9x - 6 = A(x + 2)(x - 3) + Bx(x - 3) + Cx(x + 2)$$

$$\begin{aligned} x &= 0 \Rightarrow -6A = -6 \Rightarrow A = 1 \\ x &= -2 \Rightarrow 10B = 20 \Rightarrow B = 2 \\ x &= 3 \Rightarrow 15C = -15 \Rightarrow C = -1 \end{aligned}$$ **2E1**

$$\therefore \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{1}{x} + \frac{2}{x + 2} - \frac{1}{x - 3}$$

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \int_4^6 \left(\frac{1}{x} + \frac{2}{x + 2} - \frac{1}{x - 3}\right) dx$$

$$= [\ln x + 2\ln(x + 2) - \ln(x - 3)]_4^6$$ **2E1**

$$= \left[\ln \frac{x(x + 2)^2}{(x - 3)}\right]_4^6$$

$$= \ln \frac{6 \times 64}{3} - \ln \frac{4 \times 36}{1}$$

$$= \ln \frac{2 \times 64}{4 \times 36} = \ln \frac{8}{9}$$ **1**

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5. (a)

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x + 2 & x - 2 & x + 3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$ **2E1**

(b)

$$\det A = 1 \times (2 + 1) - 0 - 1 \times 0 = 3$$ **1**

$$\det AB = x(48 - 0) - x(-48 - 0) + x(0 - 0) = 96x$$ **1**

Since $\det AB = \det A \det B$

$$\det B = \frac{\det AB}{\det A} = \frac{96x}{3} = 32x$$ **1**

6.

$$f(x) = f'(0) + xf'(0) + \tfrac{x^2}{2!}f''(0) + \tfrac{x^3}{3!}f'''(0) + \ldots$$ **1**

$$\cos x = 1 - \tfrac{x^2}{2} + \tfrac{x^4}{24} - \ldots$$ **1**

$$f(x) = \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \ldots\right)$$ **1**

$$= \tfrac{1}{2} - x^2 + \tfrac{x^4}{3} - \ldots$$ **1**

$$f(3x) = \tfrac{1}{2} - (3x)^2 + \tfrac{1}{3}(3x)^4$$

$$= \tfrac{1}{2} - 9x^2 + 27x^4 - \ldots$$ **1**

Alternative for third and fourth marks:

$$\begin{aligned} f(x) &= \tfrac{1}{2}\cos 2x & f(0) &= \tfrac{1}{2} \\ f'(x) &= -\sin 2x & f'(0) &= 0 \\ f''(x) &= -2\cos 2x & f''(0) &= -2 \\ f'''(x) &= 4\sin 2x & f'''(0) &= 0 \\ f''''(x) &= 8\cos 2x & f''''(0) &= 8 \end{aligned}$$

1

In general

$$f(x) = f(0) + f'(0)x + f''(0)\tfrac{x^2}{2} + \ldots$$

Hence

$$f(x) = \tfrac{1}{2} + 0 + (-2)\tfrac{x^2}{2} + 0 + 8\tfrac{x^4}{24} + \ldots$$

$$= \tfrac{1}{2} - x^2 + \tfrac{x^4}{3} - \ldots$$ **1**

7.

$$599 = 53 \times 11 + 16$$

$$53 = 16 \times 3 + 5$$

$$16 = 5 \times 3 + 1$$ **1**

$$\begin{aligned} 1 &= 16 - 5 \times 3 \\ &= 16 - (53 - 16 \times 3) \times 3 \\ &= 16 \times 10 - 53 \times 3 \\ &= (599 - 53 \times 11) \times 10 - 53 \times 3 \\ &= 599 \times 10 - 53 \times 113 \end{aligned}$$ **2E1**

Hence $599p + 53q = 1$ when $p = 10$ and $q = -113$. **1**

8.

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$$

Auxiliary equation: $m^2 + 6m + 9 = 0$ **1**

So $(m + 3)^2 = 0$ giving $m = -3$.

Complementary function:

$$y = (A + Bx)e^{-3x}$$ **1**

For the Particular Integral try $y = ke^{2x}$ **1**

$$\Rightarrow \frac{dy}{dx} = 2ke^{2x};\ \frac{d^2y}{dx^2} = 4ke^{2x}$$ **1**

$$4ke^{2x} + 12ke^{2x} + 9ke^{2x} = e^{2x} \Rightarrow 25k = 1$$ **1**

Hence the General Solution is:

$$y = (A + Bx)e^{-3x} + \tfrac{1}{25}e^{2x}$$ **1**

9.

$$\sum_{r=1}^{n}(4-6r) = 4\sum_{r=1}^{n} - 6\sum_{r=1}^{n} r \qquad \textbf{1M}$$

$$= 4n - 3n(n+1) \qquad \textbf{1}$$

$$= n - 3n^2$$

$$\sum_{r=1}^{2q}(4-6r) = 2q - 12q^2 \qquad \textbf{1}$$

$$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r) \qquad \textbf{1M}$$

$$= \left(2q - 12q^2\right) - \left(q - 3q^2\right) \qquad \textbf{1}$$

$$= q - 9q^2.$$

Arithmetic Series could be used, so, for the first two marks:

$$a = -2,\ d = -6 \Rightarrow S_n = \tfrac{n}{2}\{2(-2) + (n-1)(-6)\} \qquad \textbf{1}$$

$$= -2n - 3n^2 + 3n = n - 3n^2 \qquad \textbf{1}$$

10.

$$1 + x^2 = u \Rightarrow 2x\,dx = du \qquad \textbf{1}$$

$$x = 0 \Rightarrow u = 1; \qquad x = 1 \Rightarrow u = 2 \qquad \textbf{1}$$

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \int_1^2 \frac{(u-1)}{2u^4}\,du \qquad \textbf{1}$$

$$= \frac{1}{2}\int_1^2 \left(u^{-3} - u^{-4}\right)du$$

$$= \frac{1}{2}\left[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right]_1^2 \qquad \textbf{1}$$

$$= \frac{1}{2}\left[-\frac{1}{8} + \frac{1}{24}\right] - \frac{1}{2}\left[-\frac{1}{2} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{12} + \frac{1}{6}\right] = \frac{1}{24} \qquad \textbf{1}$$

The volume of revolution is given by $V = \int_a^b \pi y^2 dx$. So in this case

$$V = \pi\int_0^1 \frac{x^3}{(1+x)^4}\,dx = \frac{\pi}{24}. \qquad \textbf{1}$$

Integration by parts could be used for marks three, four and five.

$$\int_1^2 \frac{u-1}{2u^4}\,du = \frac{1}{2}\left[(u-1)\int u^{-4}du - \int 1.\frac{u^{-3}}{-3}du\right]_1^2 \qquad \textbf{1}$$

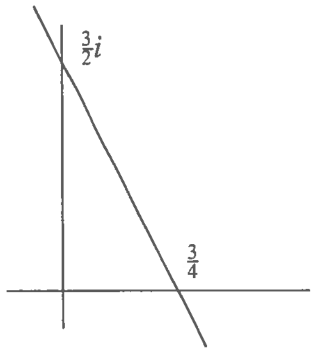
$$= \frac{1}{2}\left[\frac{u-1}{-3u^3} + \frac{u^{-2}}{(-6)}\right]_1^2 \qquad \textbf{1}$$

$$= \frac{1}{2}\left[\frac{1}{-24} - \frac{1}{24}\right] - \frac{1}{2}\left[0 - \frac{1}{6}\right]$$

$$= -\frac{1}{24} + \frac{1}{12} = \frac{1}{24} \qquad \textbf{1}$$

11.

$\lvert z - 2 \rvert = \lvert z + i \rvert$	
$\lvert (x-2) + iy \rvert = \lvert x + (y+1)i \rvert$	1
$(x-2)^2 + y^2 = x^2 + (y+1)^2$	1
$-4x + 4 = 2y + 1$	
$4x + 2y - 3 = 0$	1



1

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12. Consider $n = 1$, LHS $= (1 + a)$, RHS $= 1 + a$ so true for $n = 1$. **1**
Assume that $(1+a)^k \geqslant 1 + ka$ and consider $(1+a)^{k+1}$. **1**

$(1+a)^{k+1} = (1+a)(1+a)^k$	1
$\geqslant (1+a)(1+ka)$	1
$= 1 + a + ka + ka^2$	
$= 1 + (k+1)a + ka^2$	
$> 1 + (k+1)a$ since $ka^2 > 0$	1

as required. So since true for $n = 1$, by mathematical induction statement is true for all $n \geqslant 1$.

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13. (a)

$x = \cos 2t \Rightarrow \dfrac{dx}{dt} = -2\sin 2t;\ y = \sin 2t \Rightarrow \dfrac{dy}{dt} = 2\cos 2t$	1,1
$\dfrac{dy}{dx} = \dfrac{2\cos 2t}{-2\sin 2t} = -\cot 2t$	1
When $t = \frac{\pi}{8}$, $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\frac{dy}{dx} = -1$.	1
Equation is: $y - \dfrac{1}{\sqrt{2}} = -\left(x - \dfrac{1}{\sqrt{2}}\right)$ i.e. $x + y = \sqrt{2}$.	1

(b)

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$	**1M**
$= \dfrac{2\operatorname{cosec}^2 2t}{-2\sin 2t}$	**2E1**
$= \dfrac{-1}{\sin^3 2t}$	
$\sin 2t\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{-\sin 2t}{\sin^3 2t} + \left(\dfrac{-\cos 2t}{\sin 2t}\right)^2$	1
$= \dfrac{-1 + \cos^2 2t}{\sin^2 2t} = -1$	1

Alternative for last three marks of (b)

$$\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\operatorname{cosec}^2 2t + \left(\frac{-\cos 2t}{\sin 2t}\right)^2 \qquad 1$$

$$= \frac{-1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t} \qquad 1$$

$$= \frac{-1 + \cos^2 2t}{\sin^2 2t} = -1 \qquad 1$$

14. (a)

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

$$\int \frac{dG}{25k - G} = \int \frac{1}{25}dt \qquad 1$$

$$-\ln(25k - G) = \frac{t}{25} + C \qquad 1$$

When $t = 0, G = 0$, so $C = -\ln 25k$ 1

$$25k - G = 25ke^{-t/25}$$

$$G = 25k(1 - e^{-t/25}) \qquad 1$$

(b) When $t = 5, G = 0.6$. Therefore

$$0.6 = 25k(1 - e^{-0.2}) \qquad 1$$

$$k = 0.6/(25(1 - e^{-0.2})) \approx 0{\cdot}132 \qquad 1$$

(c) When $t = 10$

$$G \approx 3.3(1 - e^{-0.4}) \qquad 1$$

$$\approx 1.09$$

The claim seems to be justified, 1

(d) As $t \to \infty, G \to 25k \approx 3.3$ metres 1

so the limit is 3.6 metres. 1

Alternative using an Integrating Factor:

(a)

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

$$\frac{dG}{dt} + \frac{G}{25} = k \qquad 1$$

$$\text{IF} = e^{\int \frac{1}{25}dt} = e^{t/25}$$

$$\frac{d}{dt}(e^{t/25}G) = ke^{t/25}$$

$$e^{t/25}G = k\int e^{t/25}dt$$

$$= k(25e^{t/25}) + C'$$

$$G = 25k + C'e^{-t/25} \qquad 1$$

When $t = 0, G = 0$, so $C' = -25k$ 1

$$G = 25k(1 - e^{-t/25}) \qquad 1$$

15. (a) Equating the x-coordinates: $2 + s = -1 - 2t \Rightarrow s + 2t = -3$ (1)
Equating the y-coordinates: $-s = t \Rightarrow s = -t$ **1**
Substituting in (1): $-t + 2t = -3 \Rightarrow t = -3 \Rightarrow s = 3$. **1**
Putting $s = 3$ in L_1 gives $(5, -3, -1)$ and $t = -3$ in L_2, $(5, -3, -7)$.
As the z coordinates differ, L_1 and L_2 do not intersect. **1**

(b) Directions of L_1 and L_2 are: $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The vector product of these gives the direction of L_3.

$$(\mathbf{i}-\mathbf{j}-\mathbf{k}) \times (-2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2\mathbf{i}-\mathbf{j}-\mathbf{k}$$ **1M,1**

Equation of L_3:

$$\begin{aligned} \mathbf{r} &= \mathbf{i} + \mathbf{j} + 3\mathbf{k} + (-2\mathbf{i} - \mathbf{j} - \mathbf{k})u \\ &= (1 - 2u)\mathbf{i} + (1 - u)\mathbf{j} + (3 - u)\mathbf{k} \end{aligned}$$

Hence L_3 is given by $x = 1 - 2u, y = 1 - u, z = 3 - u$. **1**

(c) Solving the x and y coordinates of L_3 and L_2:

$$-1 - 2t = 1 - 2u \text{ and } t = 1 - u$$

$$\Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0$$ **1**

The point of intersection, Q, is $(-1,\ 0,\ 2)$ since $2 + 3t = 2$ and $3 - u = 2$. **1**
L_1 is $x = 2 + s, y = -s, z = 2 - s$. When $x = 1, s = -1$ and hence $y = 1$ and $z = 3$, i.e. P lies on L_1. **1**

(d) $PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. **1**

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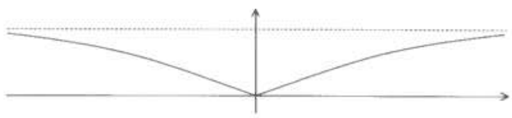
16. (a) $\tan^{-1} 2x$ has horizontal asymptotes at $y = \pm\frac{\pi}{2}$. **1,1**

(b)
$$\begin{aligned} \text{Area} &= \int_0^{1/2} \tan^{-1} 2x\, dx \quad \mathbf{1} \\ &= \int_0^{1/2} (\tan^{-1} 2x) \times 1\, dx \quad \mathbf{1} \\ &= \left[\tan^{-1} 2x \int 1.dx - \int \frac{2}{1 + 4x^2} . x\, dx\right]_0^{1/2} \\ &= \left[x \tan^{-1} 2x - \frac{1}{4}\int \frac{8x}{1 + 4x^2}\, dx\right]_0^{1/2} \\ &= \left[x \tan^{-1} 2x - \frac{1}{4} \ln(1 + 4x^2)\right]_0^{1/2} \quad \mathbf{2E1} \\ &= \left[\frac{1}{2} \tan^{-1} 1 - \frac{1}{4} \ln 2\right] - [0 - 0] \\ &= \frac{\pi}{8} - \frac{1}{4} \ln 2 \quad \mathbf{1} \end{aligned}$$

(c)



2E1

$$\begin{aligned} \int_{-1/2}^{1/2} |f(x)|\, dx &= 2\int_0^{1/2} \tan^{-1} 2x\, dx \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$ **1**

[END OF MARKING INSTRUCTIONS]