

X100/701

NATIONAL
QUALIFICATIONS
2004

FRIDAY, 21 MAY
1.00 PM - 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
3. **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions.

1. (a) Given $f(x) = \cos^2 x e^{\tan x}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'(\frac{\pi}{4})$. 3,1
 (b) Differentiate $g(x) = \frac{\tan^{-1} 2x}{1+4x^2}$. 3
2. Obtain the binomial expansion of $(a^2 - 3)^4$. 3
3. A curve is defined by the equations
 $x = 5\cos \theta, \quad y = 5\sin \theta, \quad (0 \leq \theta < 2\pi)$.
 Use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ . 2
 Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$. 3
4. Given $z = 1 + 2i$, express $z^2(z + 3)$ in the form $a + ib$. 2
 Hence, or otherwise, verify that $1 + 2i$ is a root of the equation

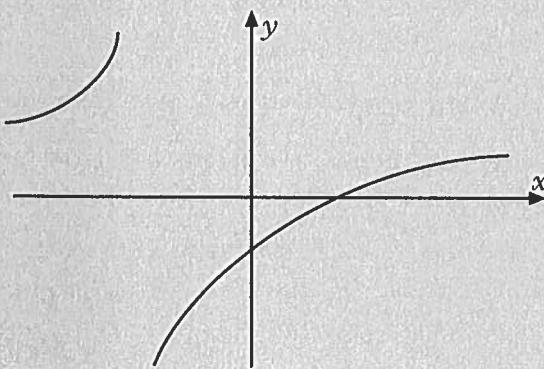
$$z^3 + 3z^2 - 5z + 25 = 0$$
. 2
 Obtain the other roots of this equation. 2
5. Express $\frac{1}{x^2 - x - 6}$ in partial fractions. 2
 Evaluate $\int_0^1 \frac{1}{x^2 - x - 6} dx$. 4
6. Write down the 2×2 matrix M_1 associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin. 2
 Write down the matrix M_2 associated with reflection in the x -axis. 1
 Evaluate $M_2 M_1$ and describe geometrically the effect of the transformation represented by $M_2 M_1$. 2
7. Obtain the first three non-zero terms in the Maclaurin expansion of $f(x) = e^x \sin x$. 5
8. Use the Euclidean algorithm to show that $(231, 17) = 1$ where (a, b) denotes the highest common factor of a and b .
 Hence find integers x and y such that $231x + 17y = 1$. 4
9. Use the substitution $x = (u - 1)^2$ to obtain $\int \frac{1}{(1+\sqrt{x})^3} dx$. 5

10. Determine whether the function $f(x) = x^4 \sin 2x$ is odd, even or neither.
Justify your answer. 3

11. A solid is formed by rotating the curve $y = e^{-2x}$ between $x = 0$ and $x = 1$ through 360° about the x -axis. Calculate the volume of the solid that is formed. 5

12. Prove by induction that $\frac{d^n}{dx^n} (xe^x) = (x + n)e^x$ for all integers $n \geq 1$. 5

13. The function f is defined by $f(x) = \frac{x-3}{x+2}$, $x \neq -2$, and the diagram shows part of its graph.



- (a) Obtain algebraically the asymptotes of the graph of f . 3
 (b) Prove that f has no stationary values. 2
 (c) Does the graph of f have any points of inflexion? Justify your answer. 2
 (d) Sketch the graph of the inverse function, f^{-1} . State the asymptotes and domain of f^{-1} . 3

14. (a) Find an equation of the plane π_1 containing the points $A(1, 0, 3)$, $B(0, 2, -1)$ and $C(1, 1, 0)$. 4

Calculate the size of the acute angle between π_1 and the plane π_2 with equation $x + y - z = 0$. 3

- (b) Find the point of intersection of plane π_2 and the line

$$\frac{x-11}{4} = \frac{y-15}{5} = \frac{z-12}{2}. \quad 3$$

[Turn over for Questions 15 and 16 on Page four]

15. (a) A mathematical biologist believes that the differential equation $x \frac{dy}{dx} - 3y = x^4$ models a process. Find the general solution of the differential equation.

5

Given that $y = 2$ when $x = 1$, find the particular solution, expressing y in terms of x .

2

- (b) The biologist subsequently decides that a better model is given by the equation $y \frac{dy}{dx} - 3x = x^4$.

Given that $y = 2$ when $x = 1$, obtain y in terms of x .

4

16. (a) Obtain the sum of the series $8 + 11 + 14 + \dots + 56$.

2

- (b) A geometric sequence of positive terms has first term 2, and the sum of the first three terms is 266. Calculate the common ratio.

3

- (c) An arithmetic sequence, A , has first term a and common difference 2, and a geometric sequence, B , has first term a and common ratio 2. The first four terms of each sequence have the same sum. Obtain the value of a .

3

- ✓ Obtain the smallest value of n such that the sum to n terms for sequence B is more than **twice** the sum to n terms for sequence A .

2

[END OF QUESTION PAPER]

Advanced Higher 2004

$$\textcircled{1} \quad (a) \quad f(x) = \cos x e^{\tan x}$$

$$f'(x) = (\cos x \cdot e^{\tan x} \cdot \sec^2 x + 2\cos x \cdot (-\sin x) e^{\tan x})$$

$$= \frac{e^{\tan x}}{e^{\tan x}} \left(1 - 2\sin x \cos x \right)$$

$$f'(\frac{\pi}{4}) = \frac{e^{\tan \frac{\pi}{4}}}{e^{\tan \frac{\pi}{4}}} (1 - \sin \frac{\pi}{2})$$

$$= 0$$

$$(b) \quad g(x) = \frac{\tan^{-1} \frac{2x}{4x^2}}{1 + 4x^2}$$

$$g'(x) = \frac{(1+4x^2) \left(\frac{1}{1+4x^2} \right)' 4 - \tan^{-1} 2x \cdot (8x)}{(1+4x^2)^2}$$

$$= \frac{2 - 8x \tan^{-1} 2x}{(1+4x^2)^2}$$

$$\textcircled{2} \quad (a^2 - 3)^4 = (a^2)^4 + 4(a^2)^3(-3)^1 + 6(a^2)^2(-3)^2 + \underline{4(a^2)(-3)^3}$$

$$= a^8 - 12a^6 + 54a^4 - 108a^2 + 81$$

$$\textcircled{3} \quad z = 1 + 2i$$

$$z^2(z+3) = \frac{5 \sin \theta}{5 \cos \theta} = \frac{5 \sin \theta}{5 \cos \theta}$$

$$= \frac{(1+2i)^2 (1+2i+3)}{(-3+4i)(4+2i)}$$

$$= \frac{-12+16i-6i+8i^2}{-12+16i-6i-8i^2}$$

$$= -20+16i$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{d\theta}{d\alpha}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \cos \theta}{-5 \sin \theta} = -\cot \theta$$

$$\tan \theta = \frac{\pi}{4}$$

$$y = 5 \cos \theta = 5 \cos \frac{\pi}{4} = \frac{5\sqrt{2}}{2}$$

$$x = 5 \sin \theta = 5 \sin \frac{\pi}{4} = \frac{5\sqrt{2}}{2}$$

Equation

$$y - \frac{5\sqrt{2}}{2} = -\left(x - \frac{5\sqrt{2}}{2}\right)$$

$$\Rightarrow 2y - 5\sqrt{2} = -2x + 5\sqrt{2}$$

$$2y + 2x = 10\sqrt{2}$$

$$\underline{y + x = 5\sqrt{2}}$$

$$z^2(z+3) = \frac{(1+2i)^2 (1+2i+3)}{(-3+4i)(4+2i)}$$

$$= \frac{(1+2i)^2 (1+2i+3)}{(-3+4i)(4+2i)}$$

$$= \frac{-12+16i-6i+8i^2}{-12+16i-6i-8i^2}$$

$$= -20+16i$$

$$\begin{aligned}
 & z^3 + 3z^2 - 5z + 25 = z(z+5) - 5z + 25 \\
 (\text{where } z = 1+i) \quad & = -20 + 10i - 5(1+i)^2 + 25 \\
 & = -20 + 10i - 5 - 10i + 25 \\
 & = 0
 \end{aligned}$$

Since $1+i$ is a root so is $1-2i$
root.

$$\begin{aligned}
 & (z - (1+i))(z - (1-2i)) \\
 & = z^2 - (1+2i)z - (1-2i)z + (1+2i)(1-2i) \\
 & = z^2 - z - 2iz - z + 2iz + 1 - 4i^2 \\
 & = z^2 - 2z + 5
 \end{aligned}$$

$$\frac{z^2 - 2z + 5}{z^3 - 2z^2 + 5z} = \frac{z^2 - 2z + 5}{5z^2 - 10z + 25}$$

$$\frac{(z+5)(z^2 - 2z + 5)}{z^2 - 5} = \frac{0}{1 \pm 2i}$$

$$\textcircled{5} \quad \text{anti-clockwise rotation } \frac{\pi}{2} \text{ radians} \quad \left(\cos \frac{\pi}{2}, -\sin \frac{\pi}{2} \right)$$

$$M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\text{reflection in } x\text{-axis } M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{so } A(x+2) + B(x-3) = 1$$

$$\begin{aligned}
 & \text{let } x = -2 \quad \text{let } x = 3 \\
 & \quad -5B = 1 \quad 5A = 1 \\
 & \quad B = -\frac{1}{5} \quad A = \frac{1}{5} \\
 & \text{so } z = 1+2i \text{ is a} \\
 & \quad \text{root.}
 \end{aligned}$$

$$\begin{aligned}
 & \text{so } \int \frac{1}{x^2 - x - 6} dx = \int \frac{1}{5(x-3)} dx + \int \frac{1}{5(x+2)} dx \\
 & \int_0^1 \frac{1}{x^2 - x - 6} dx = \int_0^1 \frac{1}{5(x-3)} dx - \int_0^1 \frac{1}{5(x+2)} dx \\
 & = \left[\frac{1}{5} \ln|x-3| - \frac{1}{5} \ln|x+2| \right]_0^1 \\
 & = \left[\frac{1}{5} \ln 2 - \frac{1}{5} \ln 3 \right] - \left[\frac{1}{5} \ln 3 - \frac{1}{5} \ln 2 \right] \\
 & = \frac{2}{5} \ln 2 - \frac{2}{5} \ln 3 \\
 & = \frac{2}{5} \ln \frac{2}{3}
 \end{aligned}$$

$$M_2 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Reflection in $y = -x$

$$f(x) = e^x \sin x$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\begin{aligned} e^x \sin x &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(x - \frac{x^3}{3!} + \dots \right) \\ &= x - \frac{x^3}{3!} + x^2 - \frac{x^4}{3!} + \frac{x^3}{2!} + \dots \\ &= x + x^2 - \frac{x^3}{3!} + \frac{x^3}{2!} + \dots \\ &= x + x^2 + \frac{x^3}{3} + \dots \end{aligned}$$

$$\begin{aligned} 231 &= 3x17 + 10 \\ 17 &= 1x10 + 7 \\ 10 &= 1x7 + 3 \\ 7 &= 2x3 + 1 \end{aligned} \Rightarrow \begin{aligned} 10 &= 231 - 3x17 \\ 7 &= 17 - 1x10 \\ 3 &= 10 - 1x7 \\ 1 &= 7 - 2x3 \end{aligned}$$

$$\begin{aligned} 1 &= 7 - 2x3 \\ &= 7 - 2x(10 - 1x17) \\ &= 2x(17 - 1x10) - 2x10 \\ &= 3x17 - 5x10 \end{aligned}$$

$$\textcircled{q} \quad x = (u-1)^2 \quad \Rightarrow \quad u = \sqrt{x+1}$$

$$\frac{dx}{du} = 2(u-1)$$

$$\frac{dx}{du} = 2(u-1) du$$

$$\begin{aligned} \int \frac{1}{(1+\sqrt{x})^3} dx &= \int \frac{1}{(1+(u-1))^3} \cdot 2(u-1) du \\ &= \int \frac{1}{(2u-2)^3} \cdot 2(u-1) du \\ &= \int 2u^{-2} - 2u^{-3} du \\ &= \frac{2u^{-1}}{-1} - \frac{2u^{-2}}{-2} + C \\ &= -\frac{2}{u} + \frac{1}{u^2} + C \\ &= -\frac{2}{\sqrt{x+1}} + \frac{1}{(\sqrt{x+1})^2} + C \\ &= \frac{1 - 2(\sqrt{x+1})^2}{(\sqrt{x+1})^2} + C \\ &= \frac{-2\sqrt{x+1}}{(\sqrt{x+1})^2} + C \end{aligned}$$

\textcircled{8}

$$(ii) f(x) = x^5 \sin 2x$$

$$\begin{aligned} f(-x) &= (-x)^5 \sin(-2x) \\ &= -x^5 \cdot -(\sin 2x) \\ &= -x^5 \sin 2x \end{aligned}$$

so $f(x)$ is an odd function
since $f(-x) = -f(x)$

$$(ii) \text{ Volume} = \int_0^1 \pi (e^{-2x})^2 dx$$

$$\begin{aligned} &= \pi \int_0^1 e^{-4x} dx \\ &= \pi \left[-\frac{1}{4} e^{-4x} \right]_0^1 \\ &= \pi \left(-\frac{1}{4} e^{-4} - \left(-\frac{1}{4} \right) \right) \\ &= \frac{\pi}{4} \left(1 - \frac{1}{e^4} \right) \end{aligned}$$

(iii) $f(x) = x^4 \sin 2x$
 $RHS = \frac{d}{dx} (xe^x) = (x+1)e^x$

Assume result is true for $n=k$
ie $\frac{d^n}{dx^n} (xe^x) = (x+k)e^x$

$$\begin{aligned} \text{Differentiate} \frac{d}{dx} RHS &\quad (xe^x) = (x+k)e^x + e^x \\ &= e^x (x+k+1) \\ &= e^x [x+(k+1)] \end{aligned}$$

so true for $n=k+1$

Hence since result is true for $n=1$ and true for $n=k \Rightarrow$ true for $n=k+1$, by induction the result is true for all $n \geq 1$.

$$(B) (a) \text{ vertical asymptote } x+2=0$$

horizontal asymptote \rightarrow divide out $x+2$ $\frac{1}{x+2} \frac{x+3}{x+2} - 5$

$$f(x) = \frac{x+3}{x+2} = 1 - \frac{5}{x+2}$$

$$x \rightarrow \infty \quad f(x) \rightarrow 1$$

so $y=1$ is a horizontal asymptote

$$(ii) \quad \frac{d^n}{dx^n} (xe^x) = (x+n)e^x$$

$$\text{For } n=1$$

$$LHS = \frac{d}{dx} (xe^x) = x.e^x + e^x$$

$$= (x+1)e^x$$

$$RHS = (x+1)e^x$$

so true for $n=1$

$$(b) \quad f'(x) = \frac{5}{(x+2)^2} > 0 \quad \text{for all } x \Rightarrow f(x) \text{ has no stationary points.}$$

so $y=1$ is a horizontal asymptote.

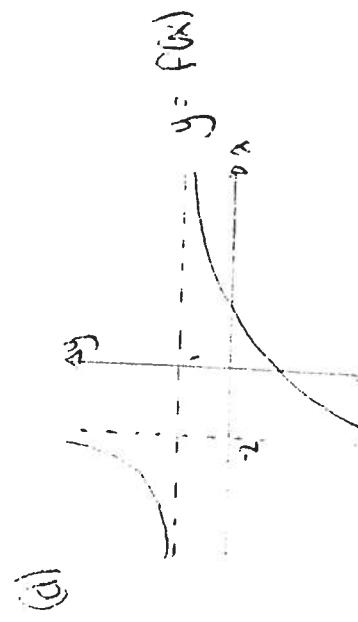
(c) Points of inflection $\Rightarrow f''(x)=0$

$$f''(x) = -10(x+2)^3$$

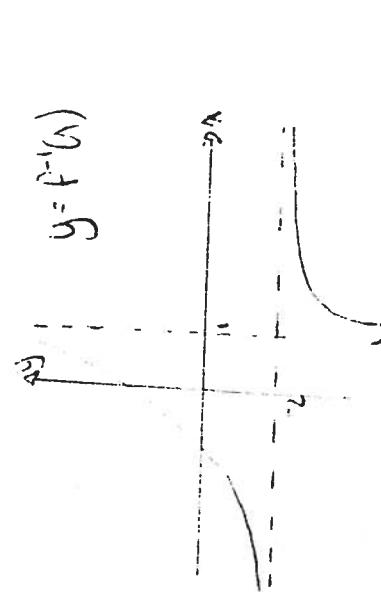
$$= -\frac{10}{(x+2)^3}$$

$$\text{If } f''(x) = 0 \Rightarrow \frac{-10}{(x+2)^3} = 0$$

not possible
 \Rightarrow no points of inflection



(d)



asymptotes $x=1$
 $y=-2$

domain $\{x \in \mathbb{R} : x \neq 1\}$.

(e)

$$\text{(i)} \quad \vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\vec{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -4 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= \frac{1}{2} (2(-4) + (-1+4) + 1(1-2))$$

$$= -2\hat{i} + 3\hat{j} - \frac{1}{2}\hat{k}$$

Equation

$$-2x + 3y - z + d = 0$$

$$\text{Substitute } A(1, 0, 3)$$

$$-\frac{1}{2} + 0 - 3 + d = 0$$

$$\text{Equation} \quad -2x + 3y - z + 5 = 0.$$

$$\text{Equation} \quad -2x + 3y - z + 5 = 0.$$

Angle between normal vectors $\underline{n}_1 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ and $\underline{n}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$|\underline{n}_1| = \sqrt{4+9+1} = \sqrt{14}$$

$$|\underline{n}_2| = -2+3+1 = 2.$$

$$|\underline{n}_1| = \sqrt{1+1+1} = \sqrt{3}.$$

$$\cos \theta = \frac{2}{\sqrt{14} \sqrt{3}} \Rightarrow \theta = 72^\circ$$

(d)

$$\begin{aligned} x &= 4t+11 \\ y &= 5t+15 \\ z &= 2t+12 \end{aligned}$$

Solutiunul ink

$$\begin{aligned} x+y-2 &= 0 \\ 4t+11+5t+15-2t-12 &= 0 \\ 7t &= -14 \\ t &= -2 \end{aligned}$$

Punt (3, 5, 8)

(a) $x \frac{dy}{dx} - 3y = x^4$

$$\frac{dy}{dx} - \frac{3}{x}y = x^3$$

Integrating factor

$$\begin{aligned} &e^{\int -\frac{3}{x} dx} \\ &= e^{\int -3x^{-1} dx} \\ &= e^{-3 \ln x} \\ &= e^{\ln x^{-3}} \\ &= x^{-3} \\ &= \frac{1}{x^3} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{x^3} y \right) &= x^3 \cdot \frac{1}{x^3} \\ \frac{1}{x^3} y &= \int dx \\ \frac{1}{x^3} dy &= x + C \end{aligned}$$

Kisi

$$\begin{aligned} x &= 4t+11 \\ y &= 5t+15 \\ z &= 2t+12 \end{aligned}$$

At $y=2, x=1 \Rightarrow$
 $c=1+c$

$$y = x^4 + cx^3$$

At $y=1, x=1 \Rightarrow$
 $2=1+c$

$y = x^4 + x^3$

(b) $y \frac{dy}{dx} - 3x = x^4$

$$y \frac{dy}{dx} = x^4 + 3x$$

$$\int y dy = \int (x^4 + 3x) dx$$

$$\begin{aligned} y^2 &= \frac{x^5}{5} + 3x^2 + C \\ y^2 &= \frac{2}{5}x^5 + 3x^2 + d \end{aligned}$$

At $y=2, x=1 \Rightarrow$
 $d=\frac{3}{5}$

$$4 = \frac{2}{5} + 3 + d$$

$$\begin{aligned} y^2 &= \frac{2}{5}x^5 + 3x^2 + \frac{3}{5} \\ y &= \pm \sqrt{\left(\frac{2}{5}x^5 + 3x^2 + \frac{3}{5}\right)^{\frac{1}{2}}} \end{aligned}$$

(iv) $S_{11} = 1 + 2 + \dots + 56$

Geometric Series

$$a = 1, r = 2, n = 56$$

$$u_n = ar^{(n-1)} \\ \bar{u}_n = 1 \cdot 2^{(n-1)} \\ 3\bar{u}_n = 3 \cdot 2^{(n-1)} \\ 3\bar{u}_{n-1} = 3 \cdot 2^{(n-2)} \\ \bar{u}_{n-1} = 2^{(n-2)}$$

$$S_n = \frac{n}{2} (u_1 + u_n) = \frac{56}{2} (1 + 2^{55}) = \underline{\underline{544}}$$

(b) $a = 2, S_3 = 24$

$$S_n = \frac{a(1 - r^n)}{1 - r} \\ 24 = \frac{2(1 - r^3)}{1 - r}$$

$$24 = \frac{2(1 - r)(1 + r + r^2)}{1 - r}$$

$$24 = 2(1 + r + r^2)$$

$$\frac{r^2 + r - 12}{r + 1} = 0 \\ (r + 4)(r - 3) = 0 \\ r = -4, 3$$

$$\underline{\underline{r = 3}}$$

(c)

Sequence A

$$a = 1, d = 2, n = 4$$

$$a = 1, r = 2, n = 4$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \\ = \frac{1(2^4 - 1)}{2 - 1} \\ = \underline{\underline{15}}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \\ = \frac{1(4^4 - 1)}{4 - 1} \\ = \underline{\underline{15}}$$

Sequence

$$S_4 = 16$$

$$S_5 = 36$$

$$S_7 = 49$$

So 1st value $n = 7$.

(d)

Sequence B

$$a = 1, r = 2, n = 4$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \\ = \frac{1(2^4 - 1)}{2 - 1} \\ = \underline{\underline{15}}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \\ = \frac{1(4^4 - 1)}{4 - 1} \\ = \underline{\underline{63}}$$

$$S_4 = 68$$

$$S_7 = 138$$

$$S_7 = 138 \frac{6}{11}$$